ART. III.—*Hydraulics of the Report on the Mississippi River of Humphreys and Abbot;* by Prof. F. A. P. BARNARD.[1]

IN former notices[2] of this important Report, we have exhibited the general scope of the work, and presented some of its interesting details in regard to the physical geography of the great hydrographic basin to which it relates. We propose, in the present article, to examine the hydraulic investigations which occupy the greater portion of its bulk, and for the sake of which the laborious survey, of which it furnishes the history and the results, was originally instituted. We are induced to do this with some particularity, because, in the first place, the report itself is not so generally accessible as could be desired; and because, secondly, its conclusions, though novel and in some respects paradoxical, are yet so convincingly established upon the broadest possible basis of experimental proof, as undoubtedly to be destined to revolutionize the whole science of river-hydraulics, if in fact they may not properly be said to have, for the first time, created it. This may be considered strong language; and, whether regarded as an encomium upon the present work or a disparagement of what has been done in the same department heretofore, it will probably be pronounced extravagant. Lest, however, we should be thought to have spoken too lightly of the labors of former observers or theorists in this department of physical investigation, we cite here, in justification, the following sketch, by a master hand, of the condition of hydraulic science, theoretical and practical, as it existed sixty years ago; and as, without the slightest substantial improvement, it has continued to exist from that time to the present. The sketch is by the eminent Dr. Robison, of Edinburgh, himself a very felicitous, though not entirely an original, writer upon the same subject. Dr. Robison, after pointing out the extent to which the interests of every civilized people are involved in questions relating to the control and distribution of the running waters of the globe, and the great variety of ways in which we are constantly engaged in endeavoring to secure such control and distribution, proceeds as follows:

"Such having been our incessant occupations with moving waters, we should expect that, while the operative artists are continually furnishing facts and experiments, the man of speculative and scientific curiosity, excited by the importance of the subject, would, ere now, have made considerable progress in the science; and that the professional engineer would be daily acting from established principle, and be seldom disappointed in

[1] Report upon the Physics and Hydraulics of the Mississippi River; upon the Protection of the Alluvial Region against overflow; and upon the Deepening of the Mouths. By Capt. A. A. HUMPHREYS and Lieut. H. L. ABBOT. Submitted to the Bureau of Topographical Engineers, War Department, 1861. 4to, pp. 456 and cxlvi.

[2] This Journal, xxxiii, 181, xxxv, 223, 234.

his expectations. Unfortunately, the reverse of this is nearly the true state of the case: each engineer is obliged to collect the greatest part of his knowledge from his own experience, and by many dear bought lessons to direct his future operations, in which he still proceeds with anxiety and hesitation: for we have not yet acquired principles of theory, and experiments have not yet been collected and published by which an empirical practice might be safely formed. * * * The motion of waters has been really so little investigated, that hydraulics may still be called a new study." And again,

"As to the uniform course of the streams which water the face of the earth, and the maxims which will certainly regulate this agreeably to our wishes, we are in a manner totally ignorant. Who can pretend to say what is the velocity of a river, of which you tell him the breadth, the depth and the declivity? Who can say what swell will be produced in different parts of its course, if a dam or weir of given dimensions be made in it, or a bridge be thrown across it; or how much its waters will be raised, by turning another stream into it, or sunk, by taking off a branch to drive a mill? Who can say with confidence what must be the dimensions or slope of this branch in order to furnish the water that is wanted; or the dimensions and slope of a canal which shall effectually drain a fenny district? Who can say what form will cause or prevent the forming of elbows, the pooling of the bed, or the deposition of sands? Yet these are the most important questions. The causes of our ignorance are the want or uncertainty of our principles; the falsity of our theory, which is belied by experience; and the small number of proper observations or experiments, and difficulty of making such as shall be serviceable."

In asserting that this extract continues to represent the state of the science of river hydraulics at the present day, as completely as it did at the close of the last century, when it was written, we rest upon the notorious fact that these very questions, and every one of these questions, which are here set forth as unsolved in the time of the writer, are as far from finding their solutions in the works of the highest authorities on the subject, at the present time, as they were then. And though Dr. Robison himself proceeds to set forth a system—theoretic and practical—mainly adopted from Dubuat, which he concludes with affirming that he has "established," and which he assures us "may be confided in as a just representation of nature's procedure;" and though many eminent laborers, since him, in the same field, have put forth other systems, each believed to be an improvement on the last; yet nothing is better known than that the opinions and practice of our ablest engineers have been, and are yet, most widely at variance in regard to the simplest problems which present themselves relating to the control and management of our natural streams; or that, just in proportion as such streams are large, or are fed by numerous tributaries, drawing their waters from regions subject to diversified climatic vicissitudes, in the same proportion they are defiant of all the dogmas of the books, and all the laws, however carefully elaborated, of hydraulic theorists. Indeed, in the very year in which the survey, of which we have in this report the results, was commenced, there was submitted to the Bureau of Topographical Engineers, at Washington, a report covering a portion of the same ground, viz: the question of the best means of preventing the overflows of the Delta of the Mississippi, by a gentleman reputed to be one of the ablest civil engineers the country has produced—the late Col. Ellet—in which all the received formulæ for determining one of the most important elements in the inquiry—the mean velocity of the stream—are set aside, and a new one introduced; and in which the conclusion is reached that protection by levees in the lower parts of the valley, is entirely impracticable. The latter view is one which many others, both before and since, have strongly held: and inasmuch as this river is a subject which has more or less occupied the mind of every man in the country having any pretensions to engineering skill, or any taste for this class of physical inquiries, it is a view which has been just as strongly discountenanced and as stoutly controverted, as it has been confidently maintained. Now if the science of river hydraulics had not been in the condition of uncertainty and imperfection which we have presumed to impute to it, how could it be possible that a great practical problem like this, the very foremost in magnitude of importance that could be stated in regard to the grandest of our rivers, could thus divide for years the opinions, not merely of the inexperts who dwell upon the banks of the stream and suffer from its ravages, but of the mathematicians and philosophers of the whole country, who exhaust, for its solution, all the resources of science; and of the practical men whose daily occupation it is to deal with precisely such cases as this, and who make the subjugation and control of unruly waters their profession!

The Mississippi has undoubtedly been, in this country, the standing opprobrium of hydraulic science. Lesser streams, in their occasional outbursts of disobedience, when they roar defiance at the artificial laws set up to govern their behavior, though they may embarrass and annoy, yet from their inferior force and volume, fail so utterly to astound the unhappy engineer whom they set at naught, as this monster of the Mississippi. Even blunders in hydraulic constructions may not glaringly betray themselves, when the force to be contended with is moderate. But the Mississippi will submit to no trifling; and the projector or the constructor who shall prescribe to it a rule which is not its own rule —that is to say, the rule of nature—will find his plans confounded or his works swept off in one wild ruin, the very first time the giant rises in his might. All this is so well known that it has become habitual with the engineers, professional or amateur, whose notions and speculations this most independent of streams so constantly disconcerts, to shift the blame of the contradiction from their own shoulders to those of the river-god himself; and to speak disparagingly, perhaps we should say bitterly, of the Mississippi, as a lawless river, a capricious river, an inconsistent river, a congeries of anomalies which will never be understood, because, like Proteus, it never presents itself twice under the same aspect.

All this depreciatory and disrespectful language, is however sadly misapplied. The Mississippi is not a lawless river: it is only a large river. It is not a capricious river; but, draining as it does an immense hydrographic basin or system of basins, its hydraulic pulsations faithfully respond to every meteorological vicissitude in all that vast region, and present therefore phenomena which, at the time and place, may not always furnish their own immediate explanation. It is not an inconsistent river; for though it may sometimes seem, to the hydraulic engineer who studies its deportment, to conform itself with a docility truly gratifying to the formulæ which he has been taught to suppose should represent its movements, and at others may contradict them in a manner the most unceremonious and the most provoking; yet it is quite an error to draw, on that account, a conclusion injurious to the character of the stream. The true mode of looking at the phenomenon is this:—The formula is indeed inconsistent with the river, but not the river with itself. For the river being one of the forms of embodied nature, if the formula fails to represent it, then the formula is inconsistent with nature: and were the river to conform itself to the formula, it would be truly an inconsistent river.

It is probably in the fact that the Mississippi is a large river that we shall find a clew to the reason that it has been pronounced lawless. The laws by which it is presumed that flowing waters are governed, though partially deduced from theory, have been modified in their expression, by the accidental peculiarities of the minor streams whose habits they were designed to represent. Reduced to formulæ, they may exhibit no wide discordance with the phenomena of those particular streams, or of others which nearly resemble them. We by no means intend to deny to them this local or limited value. But these peculiarities may so mask the action of other and more general laws, that when the same formulæ are applied to the vast volumes of water which roll along the beds of the Ganges, the Amazon or the Mississippi, where the influence of such accidents bears to the great living force of the flood a ratio much less appreciable, they may be found to fail altogether. If it is true that flowing waters are governed in their movements by determinate laws, which human observation is competent to detect, and which the human understanding is capable of comprehending, then certainly we may most reasonably hope to arrive at the discovery of those laws by attending to the examples in which their operations are illustrated upon the grandest scale. The Mississippi is therefore a river, which, so far from being lawless, is likely to afford us the most magnificent revelations of law: and its successful study is far more likely to afford us an intelligent understanding of the phenomena of lesser streams, than any amount of labor expended on such streams, to conduct us to a comprehension of the Mississippi. The correctness of this view is, we think, fully established by the Report before us; of which we will now attempt to present a succinct analysis.

The properly hydraulic portion of the Report commences with a concise exhibit of the existing state of hydraulic science, including an exhaustive catalogue of the writers who have treated of the subject. This is followed by an elaborate detail of the operations of the survey, both in the field and in the office, with a statement of the conclusions to which they successively led. Then succeeds an examination of the schemes which have been suggested for the protection of the alluvial lands of the valley against inundation; by straightening and shortening the channel of the river; by relieving it of a portion of its burthen of waters, through the diversion of tributaries or the creation of new outlets; and by confining the waters within levees or embankments higher than the highest floods. The first of these divisions we pass over. The third may form the subject of a future notice. It is the second which embraces the matter which interests us at present. In presenting the conclusions of the report as it regards the laws which govern the movements of flowing water, we shall depart from the order of the report itself so far as to group all these conclusions at once together, apart from, and in anticipation of, any explanation of the methods by which they are deduced. In this form their bearing upon each other, and their completeness as a system, will better appear. The authors, in their development of the subject, have strictly and very properly pursued the inductive method, by which the investigation was originally conducted; each conclusion being preceded by the full detail of the processes through which it was reached.

It must be premised that the problem of the flow of water in natural channels is complicated by many conditions subject to no definite law; such as the variations of cross-section, the number and magnitude of bends, and the irregularities and obstructions which may exist in the bed. In the first step toward the solution, the complications and uncertainties which the consideration of these particulars would introduce must be avoided. The laws which we are about to state, are, therefore, to be understood of waters moving uniformly in channels straight and regular. With this explanation, we cite from the report, in a form condensed from that in which we find them, the following propositions:—

In a uniformly flowing stream, the maximum velocity of the water, in any vertical plane parallel to the current, is not found at the surface, but at a point situated a little more than three tenths of the depth below the surface.

To whatever cause it may be owing, there is a resistance to the flow of water at the surface, similar in kind to that which takes place at the bottom, though usually less in degree. This resistance is propagated downward, according to a law of diminution similar to that with which the resistance at the bottom is propagated upward.

If the velocities in the same vertical plane, parallel to the current, be plotted as ordinates, and the depths at which they are observed as abscissæ, the curve drawn through the points thus determined is sensibly a parabola, having the filament of maximum velocity for its axis, which is, of course, horizontal.

This parabola varies its curvature with the changes in the mean velocity of the river; the curvature being at its maximum when the velocity is greatest. When the velocity is zero, the parabola becomes a straight line.

The law which governs the curvature is determined by the proposition, that the reciprocal of the parameter of the parabola varies as the square root of the mean velocity of the river. The reciprocal of the parameter of sub-surface velocity is therefore the ordinate in another parabola, in which the mean velocity of the river is the abscissa.

The parameter of the parabola of parameters is, for rivers generally, sensibly constant. It is, however, a function of the depth; but such a function that the variations are nearly inappreciable for river formulæ, except for depths less than twenty or perhaps twelve feet.

The variations of velocity in horizontal planes, at the surface or below it, follow the same law as in vertical planes, the curve which represents them being a parabola having its axis in the thread of maximum velocity.

The depth of the axis of sub-surface velocities in vertical planes is affected by the wind, being depressed when the wind is up-stream, and elevated when the wind is down-stream. The amount of displacement is directly proportional to the force of the wind and the depth of the river, and is sensibly the same for the same wind-force, in either direction.

Neither the velocity at the surface nor the velocity at the bottom, nor, generally, the velocity at any determinate depth, is a function of the mean velocity of the river only. The methods of gauging which have been founded on the assumption of a simple ratio, existing between some particular observed velocity and the mean velocity, are therefore all erroneous.

The ratio between the observed velocity, at any depth in any vertical plane, and the mean velocity, in the same vertical plane, is a function of three variables, which are the mean velocity of the stream, the depth of the river, and the force of the wind. There is one particular depth at which the ratio becomes independent of this last variable, and sensibly so of the depth of the river. This is the depth midway between surface and bottom.

The simplicity of the relation, between the mid-depth velocity and the mean velocity in the same vertical plane, suggests a method of gauging rivers, by which the labor of the process is greatly diminished, and its accuracy promoted. A method may be founded upon the observed velocity at *any* depth, provided the variables which affect the ratio between that and the mean velocity in the same plane are duly considered. Such a method was employed in the measurements of the Mississippi, during a considerable period of the operations of the survey.

The foregoing are the principal laws which govern the habitudes of water flowing uniformly in straight and regular channels. The following relate to the relations which exist between the cross-section, slope, and mean velocity of the stream.

The area of the cross-section, the wetted perimeter, the width, the slope, and the mean velocity of the river, are connected with each other by such relations, that, when the first three are ascertained by measurement, together with the discharge per second, the other two may be determined. For practical purposes, the wetted perimeter and the width may be treated as a single variable. The variables will then be four; and of these, if any two be given along with the discharge, unless the cross-section and mean velocity happen to be given together, the others may be found.

If a sensible addition be made to the waters of the river in any given stage, all the variables will be increased at once. The changes of the cross-section and the perimeter are, however, simple functions of the increase of depth and the inclination of the banks; or, the latter being disregarded as of trivial influence in a large river, of the increase of depth only. The change of slope is a function, but a less simple one, of the same unknown quantity. If, during a rise, the stand of the river above the extreme low water line be taken as the ordinate of a curve, and the increase of slope divided by the increase of depth be taken as the abscissa, the curve to which these coördinates correspond is a parabola. The parameter of this parabola is constant at the same locality, but its values at different localities are different.

By assuming for the change of stand, or the rise of the river, a hypothetical value, the new slope and new mean velocity may be computed; from which, in turn, a *calculated* value of the rise may be obtained. Thus, by a simple system of trial and error, the true value of the rise will easily be determined; and the resolution of involved equations avoided. In a similar manner, the depression of level, or the fall of the river, may be ascertained, in case any determinate portion of its waters be withdrawn by opening a new outlet.

The treatment of this problem by the authorities generally has the advantage over this of being greatly more simple; but it has also the disadvantage of not in the least representing nature.

These statements embrace, in brief, the substance of the contributions of this valuable report to the advancement of hydraulic science; and the basis of the new methods of practice which its authors have introduced. In estimating the effects of bends in the stream, they have adopted the principle of Dubuat, derived from observations on the flow of water in pipes, which makes the loss of living force proportional to the sum of the squares of the sines of the bending; the total amount of curvature being divided into angles below forty degrees. The agreement of the results of computation, upon this principle, with those of the observations instituted to test its correctness is very close, and is entirely satisfactory.

Whoever claims to have discovered a new law of nature, or a new truth in science, must expect the claim to be subjected to a severe scrutiny. This scrutiny will be directed equally to the processes by which he professes to have been led to it, and to the results which follow from its application in cases where deductions from it may be tested by direct observation, or by the degree of their accordance with other truths already known. This has been anticipated by the authors of this report, and they have accordingly furnished, in the amplest form, the material for applying either of the tests above suggested. We can only indicate in outline the nature of the material, referring those who would sift it thoroughly to the report itself.

Measurements of the daily discharge of the river were continuously made for periods of twelve months at Carrolton, Louisiana, of eleven months at Columbus, Kentucky, of ten months at Vicksburg, and one and a half months at Natchez. Similar observations were made upon the Arkansas at Napoleon, for eleven months; and, besides these, many others less protracted were made upon the main river and its tributaries and outlets, from the Ohio to the Gulf. These measurements required the determination of the cross-section and mean velocity at each station for every day. The cross-sections were determined by soundings made at frequent intervals, in a line at right angles to the stream. The place of each sounding was fixed by observation with two theodolites, from the extremities of a base-line of from four hundred to one thousand feet in length, measured upon the bank. Two independent sections were sounded, two hundred feet apart; and soundings repeated on the same lines, at different intervals of time, showed that the bed underwent no sensible changes. Lines of level were also run up at the banks to points above the highest floods. From these data, with the daily gauge reading, showing the stand of the river, the cross-section was known for every day.

Velocities, both at the surface and beneath it, were ascertained by means of floats. In a river of such depth and power as the Mississippi, no kind of current-meter, or other contrivance involving mechanism, is available for sub-surface observations on velocity. The submerged floats were connected with surface floats very much smaller, by means of cords. The surface floats carried small flags which were observed in their transit across two lines at right angles to the river, two hundred feet apart, by means of theodolites at the extremities of a base of the same length on the shore. The point in which each float crossed each section line was thus fixed. As many observations as possible were made at all depths, and these were as equally distributed as possible throughout the breadth of the river. The paths of the floats were then plotted and grouped, by dividing the entire width of the river into spaces or divisions, each two hundred feet wide. The mean of the velocities of all the floats in each division was taken as representing the mean velocity of that division. For the shore divisions, when the floats were not well distributed through them, a slight correction was sometimes used. These mean velocities were then multiplied by the areas of their respective divisions of the cross-section; and the sum of the products was divided by the area of the entire cross-section, for the mean velocity of the river.

The process here described was employed in all the observations of 1851, the first year of the operations of the survey. When the work, after a long suspension, was resumed in 1858, the floats were all confined to the constant depth of five feet below the surface. The mean velocities at this depth, multiplied into their corresponding division areas, gave a result called "approximate discharge," which was reduced to the true discharge by being multipled into the ratio between the velocity at this depth, and the mean velocity in the entire vertical plane parallel to the current. This ratio had not been discovered while the earlier observations were in progress.

For the determination of the law of velocities below the surface, elaborate series of observations were made at Carrolton and Baton Rouge by means of floats at different depths, from boats anchored in the stream at various distances from the shore. All the observed velocities of each set, that is, from each anchorage, were then plotted in curves. Subsequently, for the purpose of eliminating errors of observation and accidental irregularities, plots were made of the means of selected groups of observations, each group embracing the observations which corresponded to nearly equal depths and nearly equal velocities of the river. The resulting curves sufficiently indicated the existence of law, without clearly betraying its nature. Still another mode of combination of observations was resorted to. This consisted in forming a grand mean curve by taking the means of all the velocities at *proportional* instead of absolute depths—that is at every tenth, every two-tenths of depth, and so on. This was done by plotting the mean curves before obtained, on a scale which so exaggerated the differences of velocity, as to make one-thousandth of a foot an appreciable quantity, and then drawing through them parallel lines at each tenth of depth, taking the values corresponding to the points of intersection as the velocities due to those depths. Each point in this grand mean curve was fixed by two hundred and twenty-two observations: a number sufficient to obliterate, almost completely, the remaining irregularities. Its form suggests the probability that it may be one of the conic sections. In order to test this suspicion, and, in case of its truth, to determine to which of the conic sections the curve is to be referred, the general equation,

$$y^2=2Px+R^2x^2$$

is assumed; which is the equation of an ellipse when R^2 is negative, of a hyperbola when it is positive, and of a parabola when it is zero. If, in this equation, we give to x and y each two determinate values, represented by $x_{,,}$ $y_{,,}$ $x_{,,,}$ $y_{,,,}$ we may eliminate P, and obtain the value of R^2 in terms of $x_{,,}$ $y_{,,}$ $x_{,,,}$ $y_{,,,}$. As the equation assumes the origin of coördinates to be at the vertex of the curve, the plotted or tabulated velocities are not themselves the values of x required. Regarding the curve as a graphic representation of the condition of things in a vertical plane parallel to the thread of the current, it will be obvious that it must present its convexity down-stream; and that its vertex will be the point where a tangent is perpendicular to the horizon, the axis being in the line of maximum velocity. This is at about three-tenths of the depth below the surface. The abscissas will then be the differences between the maximum velocity and the velocities at other points of the curve; and the ordinates will be the distances of those points from the axis, in decimals of depth of the river. The values of R^2 as computed from these data for every tenth of depth, though never absolutely zero, are always small, and are positive in a part of the curve and negative in the rest. A mathematically regular curve was not to be expected; but the approach to a parabola is so near as to warrant the conclusion that this is the curve according to which the differences of velocity are regulated, and according to which, therefore, the resistances are distributed through the moving mass.

The equation of the parabola is easily deduced. In the equation of the common parabola, if we assume particular values, $x_{\prime\prime}$, $y_{\prime\prime}$, for the coördinates, we shall obtain the expression,

$$x = \frac{x_{\prime\prime}}{y_{\prime\prime}^2} y^2 ;$$

and if an arbitrary increase, $= x_{\prime}$, be given to all the abscissas, this will become

$$x - x_{\prime} = \frac{x_{\prime\prime} - x_{\prime}}{y_{\prime\prime}^2} y^2 ; \quad \text{or} \quad x = \frac{x_{\prime\prime} - x_{\prime}}{y_{\prime\prime}^2} y^2 + x_{\prime}.$$

In applying this equation to the observations, $x_{\prime}$ is to be replaced by the maximum velocity, and $x_{\prime\prime}$ and $y_{\prime\prime}$ by the values of those coördinates at the points most distant from the axis. The values of x are then to be computed for all the depths at which actual observations of velocity have been made. The sums of the computed and observed values are then to be compared, and their difference, if any, divided by the number of points observed, is to be applied as a correction to $x_{\prime}$ and to all the values of x: an operation which amounts to moving the whole curve slightly along the axis, without altering its curvature. By varying the depth of the axis, or the position of the point $x_{\prime\prime}$ $y_{\prime\prime}$, it may easily be found where the closest accordance between the observations and the computations can be secured.

The equation of the grand mean curve of subsurface velocities having been obtained by the processes here described, the degree of its accuracy may be tested by comparing severally the values of the velocities computed by means of it, for all the points beneath the surface at which velocities were actually observed, with the tabulated mean observed values. The comparison as made furnishes the following results: The actual maximum velocity observed being in feet 3·2611, the greatest difference found between any computed and any observed mean value is only ·0066, and the least is ·0006. The sum of all the differences, taken without regard to sign, is only ·0245, which is less than three-tenths of an inch. This test is certainly very satisfactory; but it is corroborated by others to be presently mentioned.

The forms of the parabolas of subsurface velocities at high and low water, and at intermediate stages of the river, indicated a change of curvature consequent upon a change of mean velocity. The next object was to determine the law of change. The data which presented themselves were the high water mean, the low water mean, and the grand mean curves; to which another might be added by considering that if the mean velocity of the river be made zero, the parabola becomes a straight line. An attempt was made to investigate the relation of the parameters of these curves; but the data proved to be too limited for the purpose. The idea was then conceived of attempting the same investigation in regard to the velocities in *horizontal* planes; or at the surface of the river. Data for this inquiry had been amply furnished by the observations for daily discharge. These observations, which were made at a depth of five feet below the surface, were grouped according to the even feet of approximate mean velocity of the river; and thus were obtained material for eight mean curves, corresponding to as many different mean velocities. From these was deduced a grand mean curve, as in the case of subsurface velocities. The result was a very clear disclosure of the parabolic law.

In proceeding to the study of the law of variation of curvature, equations were deduced for each of the eight mean curves. The reciprocals of the parameters of these parabolas were plotted as ordinates, the corresponding mean velocities of the river being the abscissæ—the reciprocal of the parameter of the limiting parabola, or straight line, which is zero, indicating that the curve intersects the axis of abscissæ at the origin of coördinates. A curve resulted which conformed closely to the parabolic law; and thus furnished a general expression for the reciprocal of the parameter of any parabola of surface velocities corresponding to any given mean velocity of the river. This result was tested by applying it to the formation of a general equation for the curve of velocities five feet below the surface; and employing this equation to recompute the velocities corresponding to the eight mean curves which had been used as components in forming the grand mean. The differences were all small, though larger than those in the grand mean subsurface curve; a consequence probably of the fact that each of these eight curves was founded upon a much more limited series of observations than that. A law of parameters having been thus deduced for the horizontal curves, the probability naturally suggested itself that a similar law governs those of the vertical curves also. Materials which are insufficient to reveal the existence of an unknown law are often ample for the purpose of testing the fact of its existence, after it has been once suspected. We have seen that the only materials for a curve of parameters for subsurface velocities consisted of four pairs of coördinates. Of these the parameter and mean velocity of the grand mean curve were the pair best determined, next to those of the straight line, which fixed the intersection of the curve with its axis at the origin of coördinates. A parabola, passed through the two points thus ascertained, furnished means for constructing a general equation of subsurface velocities, similar to the general equation of surface velocities before formed. The general expression for the reciprocal of the parameter, in such an equation, is, of course, the ordinate of the parameter curve corresponding to the variable mean velocity of the river (v). Or, putting b for the parameter of the curve of parameters, and $\frac{1}{2P}$ for the ordinate,

$$\left(\frac{1}{2P}\right)^2 = bv; \quad \text{or} \quad \frac{1}{2P} = \pm (bv)^{\frac{1}{2}}$$

The value of b deduced from the equations of subsurface velocities, was 0·1856.

Referring to the equation of the parabola before given, as adapted to the case in which the curve intersects the axis at a point of which the coördinates are $x = x_{\prime\prime}$, $y = 0$, viz:

$$x = x_{\prime} + \frac{x_{\prime\prime} - x_{\prime}}{y_{\prime\prime}^2} y^2,$$

it will be seen that this value of $\frac{1}{2P}$ is the coëfficient of y^2, or

$$\frac{x_{\prime\prime} - x_{\prime}}{y_{\prime\prime}^2} = -(bv)^{\frac{1}{2}},$$

the negative value being required by the nature of the case, since $x_{\prime}$ is the maximum velocity in the vertical curve. Putting then V for the general value of the velocity in the subsurface curve, and $V_{d_{\prime}}$ for the particular velocity at the depth $d_{\prime}$, which denotes the depth of the axis, the general equation of the subsurface velocities is

$$V = V_{d_{\prime}} - (bv)^{\frac{1}{2}} d_{\prime\prime}^2,$$

where $d_{\prime\prime}$ denotes the distance of the point whose velocity is V from the axis, expressed in decimals of the depth taken as unity.

This equation was subjected to a very thorough testing, by being applied to the mean high water and low water curves, and also to other series of observations not included in the formation of those curves, and to observations on the bayous of La Fourche and Plaquemine. In the mean high water curve, the mean difference between the observed and computed velocities, expressed in decimals of a foot, was ·0074, the maximum velocity in the curve being 3·8371 feet. In the mean low water curve, the same mean difference amounted to ·0127, the maximum velocity being 2·2523. This curve was not so well determined as the other, yet the mean difference is not a sixth of an inch. In medium-stage curves, deduced from 52 observations at Columbus and 20 at Vicksburg, the mean differences were ·0056 and ·0155 respectively; the maximum velocities in the curve being 4·1958 and 4·5709. In bayou Plaquemine the mean difference was ·036 on a maximum velocity of 6·491; and in bayou La Fourche ·003 on a maximum velocity of 3·250. In these two cases, the curves are deduced from the results of a single day's observation.

The same equation was also further tested by being applied to the original curves, which had been combined, as above stated, on the principle of *proportional* depths. The correctness of that principle of combination had not been quite certain; but the results of this final test left no doubt of its legitimacy. The mean discrepancy between the computed and observed velocities amounted in only one case to so much as one per cent of the maximum velocity in the curve; and in this instance, the absolute discrepancy was but a little over an inch. Usually the agreement was much nearer.

But a still more remarkable test of the accuracy of the law of parabolic velocities is furnished by a comparison of its results with observations made by Capt. Boileau at Metz, upon the flow of water in an open wooden trough only about two feet wide and one foot deep—the depth having been subsequently reduced to two-thirds of a foot. In these cases, the velocities observed in the vertical curve varied, in the first instance, from below two feet to nearly three; and in the second, from about one and a half to over two feet. The observations are recorded in the one case at fifteen different depths, and in the other at thirteen. The mean discrepancy, between these observations and the results of computation for the same points from the parabolic equation founded on them, amounted, in decimals of a foot, to only ·0330 and ·0243 for the two cases respectively. The largest of these mean errors is less than four-tenths of an inch.

It will be noticed that in all the computations made for the subsurface velocity of the Mississippi, in its different stages, and for the bayous, one and the same equation was constantly employed, viz:

$$\mathrm{V}=\mathrm{V}_{d_{\prime}}-(bv)^{\frac{1}{2}}d_{\prime\prime}{}^{2}=\mathrm{V}_{d_{\prime}}-(0{\cdot}1856v)^{\frac{1}{2}}d_{\prime\prime}{}^{2};$$

the value of $\mathrm{V}_{d_{\prime}}$ being, in each case, taken directly from the observations, and v being known by the measurements of discharge. In this proceeding it was evidently assumed that b is a constant, and always equal to 0·1856. The equations derived from Boileau's trough furnished data, not quite exact but nearly so, for obtaining a new value of b. One of the necessary data was wanting, which was the value of v, the mean velocity of the stream, but as this will not be very far in error if taken at eight-tenths of the velocity on the surface, it may be assumed to be well enough known for the purpose in hand. The numerical coëfficient of $d_{,,}^2$, in the equation of the parabola, being then divided by the so-assumed value of $v^{\frac{1}{2}}$, will give the square root of the value of b which is sought. This value is found from the equation of the trough to be a little above unity. In the equation employed in the previous computations, being that which had been derived from the grand mean curve of the Mississippi, the numerical value of b, as we have seen, was 0·1856. It appeared probable, therefore, that this quantity, that is, the parameter of the parabola of parameters, varies inversely as some function of the depth. For the sake of further testing the truth of this supposition, careful observations were made upon a feeder of the Chesapeake and Ohio Canal near Washington, having a depth of 7·1 feet and a width of 23; being, in these dimensions, much smaller than the river, and much larger than the trough. The mean maximum velocity observed was a little over two feet and a half, and the mean difference between the computed velocities for the several points observed and the means of the observations themselves at those points was less than a quarter of an inch. The equation of the parabola deduced from the observations gave a value of b equal to 0·58. The value of b (or of the parameter of the curve of parameters) changes, therefore, very slowly at considerable depths; and is practically at its minimum value for rivers when it equals 0·1856. The following expression is given as representing the observations:

$$b = \frac{1{\cdot}69}{(D+1{\cdot}5)^{\frac{1}{2}}},$$

in which D denotes the depth of the river.

It being once established, that the curve of velocities in the vertical plane parallel to the stream is a parabola, and the equation of the parabola being known, the *mean velocity in the whole vertical curve* is easily deduced. The area representing the sum of all the velocities is made up of a rectangle and a parabolic segment above the axis, and of another rectangle and parabolic segment below. Dividing the sum of these areas by the total depth will give the mean velocity required. But the dividend in this case is an expression necessarily involving the depth of the axis as an element; and, although this mean depth had been found to be nearly constant, the actual depth is observed to vary. This variation is apparently dependent on the direction and force of the wind.

In the investigation of the effects of wind-force, the selected observations were divided into three classes—those in which the wind blew up-stream, those in which it blew down-stream, and those when it either did not blow at all or blew directly across the stream. The wind-forces were estimated according to the usual scale of notation, 0 being a calm and 10 a hurricane. For the first two classes of observations, the sum of the products of the numbers of observations at each point by the force of the wind was made out for each, and the difference between the two sums, divided by the total number of observations at all the points, was presumed to give the effective force of the wind. Five sets of determinations were thus obtained, in each of which the number of observations, the depth of the axis, the mean velocity of the river, and the resultant force and direction of the wind were given.

If x be the unknown depth of the axis due to a calm, and d, $d_{,}$, $d_{,,}$, &c., the observed depths, and if y denote the amount of movement of elevation or depression of the axis which would be produced by a wind of force 1, we may presume that for movements not greater than y the changes will be sensibly proportional to the force. Also, the weight of the several wind determinations will be proportional to the number of observations from which they are deduced. Finally, the legitimate effect of an up-stream wind will be to depress the axis, and of a down-stream wind to raise it. Let the resultant wind-forces among the data be denoted by $f, f_{,}, f_{,,}$, &c., then the movements of the axis will be fy, $f_{,}y$, $f_{,,}y$, &c. Accordingly, we shall have

$$x \pm fy = d,\ x \pm f_{,}y = d_{,},\ x \pm f_{,,}y = d_{,,},\ \&c.,$$

the negative sign being used when the wind is down-stream, and the positive, when it is up. Multiplying both members of each of these equations by the number of observations from which its constants were deduced, and adding the whole, member for member, we obtain one equation containing both x and y. But if we consider that the sum of all the movements of the axis, taken without regard to sign, must be equal to the sum of all the differences between x and d, x and $d_{,}$, &c., taken positively—that is, taking $x-d$ when x is the greater, and $d-x$ when d is the greater, we shall, by multiplying once more all these differences and the corresponding movements (fy, $f_{,}y$, &c.) by the numbers of observations to which they respectively belong, and adding the products as before, obtain a second equation containing both x and y. From these equations combined, both x and y are determined. The deduced value of x is ·317 which is the depth of the axis when the wind force is zero, in decimals of the total depth of the river. The deduced value of y serves to reduce the observed depths of axis to the position of calm. A comparison of the values of x so obtained (which should all agree with each other, and with the value of x given by the equation) verifies the general correctness of the procedure; the differences found being all very slight.

In these cases, the outstanding force of wind, after balancing opposing forces, was never much above 1. Velocity observations had, moreover, been found to be impracticable, with a wind above 4. The results of the investigation, so far, would hardly therefore justify the assumption that the movement of the axis is proportional to the force, for all degrees of force. Data necessary to set at rest the doubt which here arose seemed to be wanting. A happy expedient however presented itself for the removal of the difficulty. The daily "approximate discharges" of the river had been computed for each day's observations at Columbus, Vicksburg and Natchez, by taking the sum of the products of the several division areas by the velocity observed in each, five feet below the surface. These discharges, if accurately determined, would undoubtedly vary continuously; and if plotted as ordinates to a curve of which the dates are abscissas, should present a smooth and regularly varying curvature. But if the velocity five feet below the surface is affected by the wind, we may expect to see serratures in the curve, forming prominences when the wind is down-stream, and depressions when it is up. By examining the curve at the points of serrature, the amount in cubic feet of discharge may be estimated, which, by being added or subtracted, would restore the regularity of the curve. From these data and the wind record, it is easy to see that the amount of effect on discharge due to winds of different determinate forces, as 1, 2, 3, &c., may be deduced. This amount is found to vary with the mean velocity of the river.

In applying the principle just indicated to the determination of the effect, in raising or depressing the axis, of winds of different degrees of force, observations were selected of days when the wind record showed the several forces, 1, 2, 3, and 4, up or down the river. For these days a mean cross-section, a mean approximate discharge, a mean depth or *radius* of the river, a mean approximate mean velocity and a mean velocity five feet below the surface, were computed; and also the corresponding empirical correction of discharge was deduced from the observations of the same days. The unbalanced wind force was determined in the same manner as in the former investigation.

From the mode in which the approximate discharge is computed, it is evident that—the cross-section remaining the same—this discharge may be taken as proportional to the mean velocity five feet below the surface. Hence, if the originally computed or recorded approximate discharge be increased or diminished by the product of the unbalanced force of wind into the empirical correction corresponding to that force, the sum or difference will be the approximate discharge due to a calm, and may be called the *corrected* discharge. The truth of the following proposition will then be manifest:—As the recorded discharge is to the corrected discharge, so is the observed mean velocity, five feet below the surface, to the velocity which would have been observed at the same depth had it been calm. But, had it been calm, the axis would have been at the depth ·317. In the general equation of velocities, therefore,

$$V = V_{d_{\prime}} - (bv)^{\frac{1}{2}} d_{\prime\prime}^{2},$$

if $d_{\prime\prime}$ be put for the distance from the known position of the axis to the point five feet below the surface, and the velocity found for that point by the last proportion be put for V, there will remain only one unknown quantity, which is $V_{d_{\prime\prime}}$ or the maximum velocity in the vertical curve. This velocity is therefore easily deduced, and, being substituted in the same equation, will enable us to compute the *mean* velocity in the entire vertical plane. For this mean velocity, being derived from the areas of the rectangles and parabolic segments, above and below the axis, which form the figure bounded by the curve of velocities at one end and by a vertical line at the other, the extreme length being the maximum velocity in the plane, and its breadth the depth of the river, is determined when the velocities at surface and bottom are given along with the maximum velocity and depth. The last named velocity is that which was just found; and, by the help of this, the equation gives the other two, when the proper values of $d_{\prime\prime}$, viz., distance from the axis to the surface and distance from the axis to the bottom, are substituted.

This operation was performed for each of the wind-forces, 1, 2, 3, and 4. In each case, the unbalanced wind-force of the observations was but a fraction of the total force of the wind at the given intensity. It was desirable to know the effect due to the entire force; and, in order to arrive at this, it was only necessary to reverse the operation. Thus, taking the corrected approximate discharge as the discharge due to a calm, and increasing and diminishing it by the amount of the empirical correction corresponding to each wind-force successively, we may obtain the hypothetical discharge due to the full wind-force in question, down or up the stream. Then the proportion may be stated:—As the corrected discharge is to the hypothetical discharge, so is the velocity in calm, five feet below the surface, to the velocity at the same point under the assumed wind-force. The value so determined may be substituted for V as before, or for U on the left of the more general equation following; which differs from the former only in replacing V, the velocity in a particular plane, by U which is intended to denote the velocity in the mean of all planes parallel to the axis of the river,

$$U = U_{d_{\prime}} - (bv)^{\frac{1}{2}} d_{\prime\prime}^{2} = U_{d_{\prime}} - (0{\cdot}1856v)^{\frac{1}{2}} d_{\prime\prime}^{2}.$$

In this equation, $U_{d_{\prime}}$ and $d_{\prime\prime}$ are both unknown. The equation for *mean velocity in the vertical plane* contains (as above stated) values of U at the surface and at the bottom, both of which are expressible, from the equation just given, in terms containing only the same two unknowns, $U_{d_{\prime}}$ and $d_{\prime\prime}$.[2] There will therefore be two independent equations, involving only two unknowns, and from these the depth of the axis, on which the values of $d_{\prime\prime}$ depend, may be determined.

This determination having been made for the four wind-forces independently, the amount of displacement of the axis due to each wind-force is ascertained. The result is, that the displacement is directly proportioned to the wind-force, that it is equal for the same wind-force in opposite directions, but that it is a depression for an up-stream wind and an elevation for a down-stream wind. It also appears that the amount of displacement is independent of the mean velocity of the river. And although, in this investigation, the data for determining the effect of each wind-force are entirely independent of those for the others, yet the results exhibit a remarkably close agreement. As a result of the whole we obtain the following formula which is a general expression for the depth of the axis, (denoted by $d_{\prime\prime}$) f representing the force of the wind, and r the radius, or mean depth of the river:—

$$d_{\prime}=(0{\cdot}317+0{\cdot}06f)r.$$

Since the process just described furnishes the means of expressing the velocity at the surface or at any depth below it, in terms containing all the variables which affect its value, it is manifestly practicable to deduce a system of gauging a river in which the true discharge shall be obtained from observations at one unvarying depth. But, as the ratio of the so observed velocity to the mean velocity is a varying one, no system of gauging founded on this method of observation can be relied on, in which account is not taken of this variation.

There is, however, one velocity—the velocity at mid-depth—which bears a ratio to the mean velocity in the vertical plane which is nearly constant. In order to show the evidence of this, we must actually state the equation for this mean velocity to which we have several times referred. It will be necessary therefore to adopt the following symbols:

[2] The symbol $d_{\prime\prime}$ has, apparently, two distinct values as employed above:—it being used to stand both for the distance from the axis to the surface, and for the distance from the axis to the bottom. But, as the depth is known, these are simple functions of a single unknown quantity. We have endeavored, in the explanation of the principles of the report, to avoid the introduction of many symbols. We have not therefore, thus far, actually stated the equation for *mean velocity in the vertical plane* above referred to. The system of symbols employed in the report is very clear and expressive; but we have chosen to defer an account of it, until after completing our outline of principles and processes.

D=depth of river. d=distance below the surface (variable).
$d_{,}$=depth of axis (line of maximum velocity in the vertical plane).
m=depth of line of mean velocity in the vertical plane.
V=velocity at any point in the vertical plane.
$V_{d_{,}}$=velocity at the axis, or maximum velocity.
V_m=mean velocity in the vertical plane.
V_0=velocity at the surface. V_D=velocity at the bottom.

Then $V_m D$ will be the value of an area equal to the two rectangles and two parabolic segments concerned in the determination of the mean velocity. The truth of the following equation is therefore manifest:—

$$V_m D = \tfrac{2}{3}(V_{d_{,}} - V_0)d_{,} + V_0 d_{,} + \tfrac{2}{3}(V_{d_{,}} - V_D)(D - d_{,}) + V_D(D - d_{,}).$$

Which, reduced, becomes,

$$V_m = \tfrac{2}{3}V_{d_{,}} + \tfrac{1}{3}V_D + \tfrac{1}{3}\frac{d_{,}}{D}(V_0 - V_D).$$

Take now the general equation for the vertical plane, heretofore given,

$$V = V_{d_{,}} - (bv)^{\frac{1}{2}} d_{,,}^2 = V_{d_{,}} - (bv)^{\frac{1}{2}}\left(\frac{d - d_{,}}{D}\right)^2$$

and substitute in it the value of d at the surface, $=0$, and at the bottom, $=D$, and we have the two expressions following:—

$$V_0 = V_{d_{,}} - (bv)^{\frac{1}{2}}\left(\frac{d_{,}}{D}\right)^2. \quad V_D = V_{d_{,}} - (bv)^{\frac{1}{2}}\left(\frac{D - d_{,}}{D}\right)^2.$$

These values of V_0 and V_D being introduced into the expression for the value of V_m, we shall have, after reduction and transposition,

$$V_{d_{,}} = V_m + (bv)^{\frac{1}{2}}\left(\tfrac{1}{3} + \frac{d_{,}(d_{,} - D)}{D^2}\right).$$

And substituting this value of $V_{d_{,}}$ in the foregoing general equation for velocity in the vertical plane, there will result after reduction,

$$V = V_m + (bv)^{\frac{1}{2}}\left(\frac{D^2 - 3d_{,}D - 3d^2 + 6dd_{,}}{3D^2}\right).$$

Divide the identical equation, $V_m = V_m$, member for member, by this expression, and we obtain finally the general ratio of the velocity at any depth to the mean velocity in the vertical plane, viz:

$$\frac{V_m}{V} = \frac{V_m}{V_m + (bv)^{\frac{1}{2}}\left(\frac{D^2 - 3d_{,}D - 3d^2 + 6dd_{,}}{3D^2}\right)},$$

in which $d_{,}$ must be replaced by its value as deduced from the investigation of the effect of wind-force, in order that all the variables may explicitly appear.

If any value could be assigned to d in this expression, which should reduce the fraction in brackets to zero, it would follow that the depth m (which would then be d) is independent of the mean velocity of the river, v. This cannot be done; but if d be made $=\frac{1}{2}$D, the ratio is greatly simplified, and the equation becomes

$$\frac{V_m}{V_{\frac{1}{2}D}}=\frac{V_m}{V_m+\frac{1}{12}(bv)^{\frac{1}{2}}}.$$

Since both D and d, disappear from this expression, the ratio of the mid-depth velocity is independent both of the depth of the river and of the force of the wind. If the velocities in the mean of all vertical planes parallel to the current be represented by U instead of V, we shall have

$$\frac{U_m}{U_m+\frac{1}{12}(bv)^{\frac{1}{2}}},$$

in which, in a channel of rectangular cross-section, U_m would be equal to v. In point of fact, for the Mississippi, it is equal to $\cdot 93v$, the coëfficient remaining sensibly constant. Substituting this value, the authors of the report have tested the formula, by computing the ratio,

$$\frac{\cdot 93v}{\cdot 93v+\frac{1}{12}(bv)^{\frac{1}{2}}},$$

for every even foot of velocity from 1 to 8, and employing the results in the computation of mean velocities from many mid-depth velocities actually observed. The observations include a number on the Mississippi and the outlet bayous, in different stages of the water, and also those made, as before mentioned, on the feeder of the Chesapeake and Ohio canal, together with others by Messrs. Hennocque and Defontaine on the Rhine, and finally those by Mr. Boileau on his experimental wooden trough. The position of the axis, among these data, varied from the surface to a point below mid-depth, and the mean velocities varied from a foot and a half to more than four feet. The differences between the observed and computed values of $\frac{V_m}{V_{\frac{1}{2}D}}$, however, were, for the most part, practically insensible, and in the few cases in which this was not true they amounted to but two or three per cent.

The near approach to constancy, and equality of the ratio between the mid-depth velocity and the mean velocity, is easily intelligible when the fact is once detected. The resistances to motion proceed from the perimeter; that is, from the bottom and the surface. The discharge remaining sensibly constant, and, in a uniform channel, the cross-section also, whatever retards the movement at one surface necessitates an acceleration at the other.

An up-stream wind, by diminishing the velocity at the surface, creates a slight increase of slope; and this would increase the mean velocity of the stream, but for the fact that, as much as the new slope would add, the wind destroys, so that the mean velocity remains constant. What the wind destroys at the surface must therefore be compensated at the bottom; and, as these effects are propagated through the mass according to the same laws as the ordinary resistances, the curve of velocities continues to be parabolic, but the level of its axis is changed.

The simplicity of this ratio suggests an improved method of gauging streams; but into these practical details it is not important that we should enter here.

We may here conveniently present, in a single group, the most important of the formulæ which result from the investigations of which we have been endeavoring to give an account.

Those which have not been fully explained, are easily deducible from those which have.

$$V=V_{d_{\prime}}-(bv)^{\frac{1}{2}}\left(\frac{d-d_{\prime}}{D}\right)^{2}. \quad V_{0}=V_{d_{\prime}}-(bv)^{\frac{1}{2}}\left(\frac{d_{\prime}}{D}\right)^{2},$$

$$V_{D}=V_{d_{\prime}}-(bv)^{\frac{1}{2}}\left(1-\frac{d_{\prime}}{D}\right)^{2}. \quad V_{\frac{1}{2}D}=V_{m}+\frac{1}{12}(bv)^{\frac{1}{2}}.$$

$$V_{m}=\frac{2}{3}V_{d_{\prime}}+\frac{1}{3}V_{D}+\frac{1d_{\prime}}{3D}(V_{0}-V_{D}).$$

$$V_{d_{\prime}}=V_{m}+(bv)^{\frac{1}{2}}\left(\frac{1}{3}+\frac{d_{\prime}(d_{\prime}-D)}{D^{2}}\right).$$

$$V=V_{m}+(bv)^{\frac{1}{2}}\left(\frac{D(\frac{1}{3}D-d_{\prime})+d(2d_{\prime}-d)}{D^{2}}\right).$$

$$d_{\prime}=(0{\cdot}317+0{\cdot}06f)r. \quad U_{m}=0{\cdot}93v.$$

$$U_{0}=0{\cdot}93v+(0{\cdot}016-0{\cdot}06f)\,(bv)^{\frac{1}{2}}.$$

$$U_{r}=0{\cdot}93v+(0{\cdot}06f-0{\cdot}350)(bv)^{\frac{1}{2}}.$$

$$U_{d_{\prime}}=0{\cdot}93v+\left((0{\cdot}317+0{\cdot}06f)^{2}-0{\cdot}06f+0{\cdot}016\right)(bv)^{\frac{1}{2}}.$$

$$U=0{\cdot}93v+\left(\frac{dr(0{\cdot}634+0{\cdot}12f)-d^{2}}{r^{2}}-0{\cdot}06f+0{\cdot}016\right)(bv)^{\frac{1}{2}}.$$

$$v=\left((1{\cdot}08U_{\frac{1}{2}r}+0{\cdot}002b)^{\frac{1}{2}}-0{\cdot}045b^{\frac{1}{2}}\right)^{2}.$$

(To be continued.)