ART. XXXVI.—*Studies in the Electro-magnetic Theory.—I. The law of electro-magnetic flux;** by M. I. PUPIN, Ph.D., Columbia College, New York.

THE law of electro-magnetic flux is a short expression for the well-known quantitative relations between electromotive force and electric flux on the one hand and magnetomotive force and magnetic flux on the other. Ohm's law is a part of it.

These relations are statements of experimental facts which we know to hold true for constant and slowly varying forces. The object of this investigation is to show the exact position which this law occupies in Maxwell's electro-magnetic theory; to point out its limitations; to show that Maxwell's electro-magnetic theory of light demands a more general form of this law; and finally, to present a general form of this law of which both its ordinary form and also those forms which were assumed hypothetically in some of the recent developments of the electro-magnetic theory of light are special cases.

1. *The two fundamental laws of Maxwell's Electro-dynamics.*

A brief statement of Maxwell's electro-magnetic theory seems desirable in this discussion. For the sake of brevity none but its most essential features will be presented, and that in such a way as to emphasize as forcibly as possible the essential differences between this theory and the older electro-magnetic theories.

The essential features of the Maxwellian theory are reducible to two, which may be called its two characteristic features. These two characteristic features can be exhibited in a very simple manner by considering the gradual change in form and in meaning which the following two well known experimental laws, relating to magneto-electric and to electro-magnetic induction, undergo as we pass from the views of old electric theories to those of the Maxwellian theory. These laws I choose to state in the following form for reasons which will be evident presently:—

First law:—A varying magnetic field induces a field of electric force in all electric conductors within its region. The electromotive force around any simple circuit† of this induced

* Read in abstract before the Am. Assoc. Adv. Science at its Springfield meeting, Aug. 30, 1895.

† The expression *simple circuit* needs an explanation. Consider any point of the field. Pass a plane through it and in this plane draw an infinitely small area around the point under consideration. If the boundary of this elementary area be such that none of its points contain more than one branch of the boundary curve, then this boundary curve is a simple circuit around this point.

electric field is proportional to the rate of variation of the magnetic flux through any surface bounded by this circuit.

Second law:—An electric current induces a field of magnetic force . . . . The magnetomotive force around any simple circuit of this induced magnetic field is proportional to the current passing through any surface bounded by this circuit.

For the sake of brevity the factors of proportionality are not explicitly mentioned. It is well to observe, however, that they are numerical and that by a suitable selection of electric units they could be made unity in each case.

There is evidently a considerable formal resemblance between these two experimental laws when they are stated in the form in which they have just been stated. This formal resemblance can be carried considerably further without departing from the views of pre-Maxwellian electric theories. All that is needed is a simple adjustment of the electric terminology of the pre-Maxwellian period. This is the step which is now in order. It leads along the shortest path from old theories to Maxwell's electro-magnetic theory.

Consider two electric conductors A and B, insulated from each other, one charged with a certain quantity of positive electricity and the other with an equal charge of negative electricity. Consider now a surface surrounding any one of these two conductors. The total electric flux through this surface is numerically equal to the total quantity of electricity enclosed by the surface. Hence it is also equal to the total *integral electric current* which would be obtained if the total positive charge were carried across this surface to the negatively charged conductor and thus the two conductors reduced to their neutral state. We can speak, therefore, of a total electric flux across a surface as of a *fictitious integral current* through this surface. We can speak of it also as of a *total electric displacement* in the sense that if the two conductors are neutral and a certain quantity of positive or negative electrification is transferred from one to the other and therefore displaced from one side of the bounding surface to the other we shall have a total electric flux across this surface numerically equal to the total electric displacement across the same surface. With this mental reservation, namely, that we are using certain terms in their figurative sense only, we can always employ the expressions "*total electric flux*," "*total integral current*," and "*total electric displacement*" as synonymous terms without departing from the views of pre-Maxwellian theories.

So far we have not restricted the fictitious total integral current or displacement to any particular paths. That shall be done now.

Let these paths be such that at any moment the total fictitious electric transference through any elementary area up to that moment is equal to the electric flux through that area at that moment. Then, as long as we remember that we are speaking of a *fictitious* "*integral current*" and a *fictitious* "*electric displacement*" through any elementary area in an electric field, we can employ these two terms as synonymous with "electric flux" through that area without departing from the views of the pre-Maxwellian period.

Again just as the electric current through any elementary area of a conductor is defined as the rate of variation of the integral current, so we can also speak of a current through the dielectric without deserting the views of old electro-magnetic theories provided that we take it as granted that this dielectric current is fictitious, since it is the rate of variation of a fictitious integral current. The expression "*dielectric or displacement current*" becomes, therefore, synonymous with rate of variation of the electric flux. Similarly we can substitute the expression *magnetic current* for the expression "*rate of variation of magnetic flux.*"

To distinguish the real, that is the electric conduction current, from the fictitious or displacement current, the expression "*conduction current*" must be used when the real and not the fictitious (displacement) current is meant.

This precaution is unnecessary in the case of the magnetic current, since there is no magnetic conduction current.

The laws of magneto-electric and of electro-magnetic induction can now be stated more symmetrically, as follows:—

First law:—A region of magnetic . . . . . currents induces a field of electric force *in all electric conductors within that region.* The electromotive force around any simple circuit in this induced electric field is proportional to the magnetic . . . . . . current passing through any area which is bounded by this circuit.

Second law:—A region of electric *conduction* currents induces a field of magnetic force . . . . The magnetomotive force around any simple circuit in this induced magnetic field is proportional to the electric *conduction* current passing through any area which is bounded by this circuit.

The formal resemblance between the two laws is very striking now. It would be perfect if we either omitted the words which are in italics or filled out suitably the dotted lacunæ.

The last alternative is not admissible, because we know nothing about magnetic conduction currents, nor about magnetic conductors. The first alternative does not strike one so unfavorably. There is really no evidence against the permissibility of omitting the words in italics from the statement of the first law. It is true that Faraday's experiments, by which he discovered that law, prove the existence of no other induced electromotive forces excepting those which Faraday detected in conductors. But these experiments neither affirm nor do they deny the presence of magneto-electric induction in dielectrics. On the other hand, Faraday's experiments on dielectric and diamagnetic substances and his speculative views of electro-magnetic phenomena urge us with an irresistible force to the belief that electromotive forces just like magnetomotive forces are induced around every circuit, no matter whether that circuit pass through a conductor or through a dielectric (including the most perfect of all dielectrics, that is a perfect vacuum), and that just as the magnetic displacement current is real in the sense that it produces inductive effects, so the electric displacement current is not a fiction, as one who is faithful to the views of older electric theories has to assume, but it is just as real as the electric conduction current in the sense that it is an actually existing process in the dielectric, which is just as capable of inducing magnetomotive forces as the conduction current. The mechanism of this process is, of course, just as unknown to us as the mechanism of that process which is called the electric conduction current. Maxwell was the first to feel the force of this tendency of Faraday's experiments and speculations and to yield to it, making thus a radical departure from the views of old electric theories. His statement of the two laws of induction omits, therefore, the words which are in italics in the last statement of these laws. Hence the following wording of these two laws is in accordance with Maxwell's views:—

First law:—*Every magnetic current induces a field of electric force. The electromotive force around any simple circuit in this induced electric field is proportional to the magnetic current passing through any area which is bounded by this circuit.*

Second law:—*Every electric current induces a field of magnetic force. The magnetomotive force around any simple circuit in this induced magnetic field is proportional to the electric current passing through any area which is bounded by this circuit.*

In this generalized form these two laws form the foundation of Maxwell's electro-magnetic theory.* This theory may, therefore, be described broadly as that theory which generalizes the two experimental laws of magneto-electric and of electro-magnetic induction by extending the region of magneto-electric induction and of the electric current from conductors to the dielectric.

* See O. Heaviside, Phil. Mag., February, 1888; H. Hertz, Wied. Ann., xxiii, p. 84, 1884; xl, p. 577, 1890.

So far an electro-magnetic field of invariable geometrical configuration has been considered. It should be observed now that the two laws suffice to describe completely the inductive effects in an electro-magnetic field of variable geometrical configuration also. A field of variable geometrical configuration means, of course, a field in which the various *sources of flux* (like electrically or magnetically charged bodies and conductors carrying electric currents) are in relative motion with respect to bodies which are under the inductive action of these sources. According to these laws the inductive effects in any circuit depend on the rate of variation of the flux through that circuit and on nothing else; they are, therefore, independent of the particular method by which that variation is produced; whether by varying the intensity of the field; or by motion of the various parts of the field; or by keeping everything constant and moving the circuit under consideration, it is immaterial. This is simply an extension of the experimental fact that the electromotive force induced in a conducting circuit follows the same law whether that electromotive force be induced by the motion of the circuit towards a magnet or by the motion of the magnet towards the circuit, or by keeping the two in fixed relative position and varying the strength of the magnet.

The fundamental quantities in this theory are electric current and electromotive force on the one hand and magnetic current and magnetomotive force on the other. Calling them the *fundamental vectors of the electro-magnetic field*, we can say that the *first characteristic feature of Maxwell's electro-magnetic theory consists in the perfectly symmetrical form of cross-connection between its fundamental vectors, one of these cross-connections stating the law of induced magnetomotive force and the other stating the law of induced electromotive force.*

A symbolical statement of these two laws shows this symmetry more clearly.

Let X, Y, Z be the components of the induced electromotive intensity at any point of the field.

Let α, β, γ be the components of the induced magnetomotive intensity at same point of the field.

Let $\left.\begin{array}{l}\frac{df}{dt}, \frac{dg}{dt}, \frac{dh}{dt}, \\ \frac{da}{dt}, \frac{db}{dt}, \frac{dc}{dt},\end{array}\right\}$ be the components of the intensity of the electric and magnetic currents, respectively, at the same point.

If the electric vectors be measured in electrostatic and the magnetic vectors be measured in electro-magnetic units, then, V being the ratio between the two, we can state symbolically the two laws as follows:

$$\left.\begin{aligned}\frac{4\pi}{V}\frac{da}{dt} &= -\left(\frac{\delta Z}{\delta y}-\frac{\delta Y}{\delta z}\right)\\ \frac{4\pi}{V}\frac{db}{dt} &= -\left(\frac{\delta X}{\delta z}-\frac{\delta Z}{\delta x}\right)\\ \frac{4\pi}{V}\frac{dc}{dt} &= -\left(\frac{\delta Y}{\delta x}-\frac{\delta X}{\delta y}\right)\end{aligned}\right\}\text{ First law.}$$

$$\left.\begin{aligned}\frac{4\pi}{V}\frac{df}{dt} &= \frac{\delta\gamma}{\delta y}-\frac{\delta\beta}{\delta z}\\ \frac{4\pi}{V}\frac{dg}{dt} &= \frac{\delta\alpha}{\delta z}-\frac{\delta\gamma}{\delta x}\\ \frac{4\pi}{V}\frac{dh}{dt} &= \frac{\delta\beta}{\delta x}-\frac{\delta\alpha}{\delta y}\end{aligned}\right\}\text{ Second Law.}$$

An immediate consequence of these two laws is the following important relation:—

$$\frac{\delta}{\delta x}\left(\frac{da}{dt}\right)+\frac{\delta}{\delta y}\left(\frac{db}{dt}\right)+\frac{\delta}{\delta z}\left(\frac{dc}{dt}\right)=0$$

$$\frac{\delta}{\delta x}\left(\frac{df}{dt}\right)+\frac{\delta}{\delta y}\left(\frac{dg}{dt}\right)+\frac{\delta}{\delta z}\left(\frac{dh}{dt}\right)=0.$$

That is to say, the magnetic and the electric currents follow the laws of flow of an incompressible fluid. Hence they form closed paths. This is one of the essential differences between the Maxwellian theory and the older electro-magnetic theories. Numerous other essential differences can be deduced directly from the two laws by a proper interpretation of the physical meaning of the perfect formal similarity existing between them. One or two only of the several of these essential differences which were first pointed out clearly by Hertz,* should be mentioned here, in order to show their general character. The magnetic field induced by a loop of wire carrying an electric current (which we may call a varying electric solenoid) has the same form as the electric field induced by a varying magnetic solenoid which has the same geometrical configuration as the loop; hence just as two varying electric solenoids exert a magnetic force, so two varying magnetic solenoids exert an electric force upon each other.

An experimental proof of the existence of the electric field of force induced by a varying magnetic solenoid would constitute a *direct experimental evidence* in favor of the Maxwellian theory. It is evident, however, that extremely serious and apparently insurmountable technical difficulties would be encountered by every attempt to verify experimentally this relation and similar other reciprocal relations which, taken

* Wied. Ann. xxiii, p. 84, 1884.

together, constitute the physical meaning of the perfect similarity in form between the two fundamental laws of Maxwell's electro-magnetic theory.* Rowland's classical experiment which proved the existence of a magnetic field produced by a moving electrostatic charge, and Röntgen's detection of the magnetic effect of a varying dielectric polarization, may be considered as the only direct experimental evidences ever obtained.

2. *Second part of Maxwell's theory.*

A search for a source of indirect experimental evidences leads to the second and most interesting, but at the same time most difficult, part of Maxwell's electro-magnetic theory. *The principal feature of this part is the second characteristic feature of this theory. It is the importance of the function which is performed by the physical constants of the field, namely, by the specific inductive capacity, the magnetic permeability, and the resistivity.*

These constants Maxwell defines by the law of flux, which in the well known notation is expressed as follows:

$$\left.\begin{aligned} f &= \frac{K}{4\pi}X,\ g = \frac{K}{4\pi}Y,\ h = \frac{K}{4\pi}Z \\ a &= \frac{\mu}{4\pi}\alpha,\ b = \frac{\mu}{4\pi}\beta,\ c = \frac{\mu}{4\pi}\gamma \\ u &= \frac{X}{k},\ v = \frac{Y}{k},\ w = \frac{Z}{k} \end{aligned}\right\}\ \text{for isotropic media.}$$

$$\left.\begin{aligned} f &= \frac{K_1}{4\pi}X,\ g = \frac{K_2}{4\pi}Y,\ h = \frac{K_3}{4\pi}Z \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ u &= \frac{X}{k_1},\ v = \frac{Y}{k_2},\ w = \frac{Z}{k_3} \end{aligned}\right\}\ \text{for anisotropic media.}$$

It should be observed now that the first part of this law, referring to isotropic media, is a statement of experimental facts as long as we confine ourselves to constant and slowly-varying forces; it is an hypothesis† in every other case. The second part, referring to anisotropic media, was a pure hypothesis at the time when Maxwell first stated it. He justified it by pointing out the close connection between the optical and the electro-magnetic constants. Hence just as double refraction of light in a substance proves that the optical constants of the substance are different in different directions, so the double refraction of

* Lodge, Phil. Mag., June, 1889.

† See Goldhammer, Wied. Ann. xlvi, p. 99, 1892.

electro-magnetic waves in a substance can be accounted for by the electro-magnetic theory by assuming that the electro-magnetic constants of the substance are different in different directions. There seems to be no strong evidence against the assumption that Maxwell looked upon this form of the law of flux as merely a first approximation to truth.

We are ready now to describe the sources of indirect experimental evidences in favor of the hypotheses on which rest the two fundamental laws of Maxwell's electro-dynamics. It seems desirable, however, to make first the following slight digression. Combining the two fundamental laws with the law of flux and considering, for the sake of brevity, that the medium is isotropic and non-absorptive, we obtain:

$$\left.\begin{array}{l} \frac{\mathrm{K}}{\mathrm{V}}\frac{d\mathrm{X}}{dt} = \frac{\delta\gamma}{\delta y} - \frac{\delta\beta}{\delta z} \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \end{array}\right\} (\mathrm{A})$$

$$\left.\begin{array}{l} \frac{\mu}{\mathrm{V}}\frac{\delta\alpha}{dt} = -\left(\frac{\delta\mathrm{Z}}{\delta y} - \frac{\delta\mathrm{Y}}{\delta z}\right) \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \end{array}\right\} (\mathrm{B}).$$

Equations (A) and (B) and similar equations for absorptive and anisotropic media are very often called the mathematical statement of the two fundamental laws of Maxwell's electro-dynamics. Such a presentation of the Maxwellian theory seems not only entirely unnecessary but also misleading. For the hypotheses by means of which Maxwell passed from the two experimental laws of magneto-electric and of electro-magnetic induction to the two fundamental laws of his electro-dynamics have very little if anything to do with what may or may not be the complete form of the law of flux as stated above, not even if we choose to accept Maxwell's mechanical models of the electro-magnetic field as correct representations of what is going on in such a field. In these models the constants K, μ, k, have a definite mechanical meaning, but this meaning does by no means lead to the conclusion that for all periodicities of the impressed forces equations (A) and (B) are equivalent to that mathematical statement of the two fundamental laws in which the current and the forces appear as fundamental vectors, and no reference whatever is made to the physical constants of the field. Equations (A) and (B) should, therefore, be treated as an adulteration of Maxwell's fundamental laws, the foreign element in this adulteration being that form of the law of flux which Maxwell employed. This adulteration may do, and as will be pointed out presently, actually does just as well, in many instances, as the real thing, but there are considerations, to be discussed presently, which seem to speak very forcibly against the equivalence between this adulteration and the real thing.

The main thread of the discussion can now be taken up again. Combining the two fundamental laws with the law of flux, the well known relations of which the following are types are easily obtained:

$$\frac{d^2X}{dt^2} + \frac{4\pi}{k_1 K_1}\frac{dX}{dt} = \frac{V^2}{\mu K_1}\Delta^2 X$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{d^2\alpha}{dt^2} + \frac{4\pi}{k_1 K_1}\frac{d\alpha}{dt} = \frac{V^2}{\mu K_1}\Delta^2 \alpha$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

For all non-absorptive isotropic insulators we shall have

$$\frac{d^2X}{dt^2} = \frac{V^2}{K}\Delta^2 X$$

$$\cdots\cdots\cdots$$

$$\frac{d^2\alpha}{dt^2} = \frac{V^2}{K}\Delta^2 \alpha$$

$$\cdots\cdots\cdots$$

These equations describe the source of indirect experimental evidences of the Maxwellian theory. They constitute the second part of this theory, and they show better than any words can the important function which the physical constants of the field perform in our modern views of electro-magnetic phenomena. It was reserved for Hertz and his followers to reveal by actual experiment the wealth of this source and to demonstrate beyond all reasonable doubt that long electro-magnetic waves are propagated in perfect accordance with the laws stated in these equations. These laws are, within certain limits which will be mentioned presently, a perfect analogy to the laws of propagation of luminous waves. Hence experimental results obtained from electro-magnetic phenomena involving the operation of forces whose period is not shorter than the shortest period of the Hertzian waves, give a decisive victory for the hypotheses which underlie the Maxwellian electro-magnetic theory.

But this theory is much broader. It states that, probably, light is an electro-magnetic phenomenon. It fails, however, to give a satisfactory account of some of the most striking optical phenomena, as, for instance, dispersion, absorption, rotation of the plane of polarization, etc. In other words, it fails to give a satisfactory general relation between optical and electro-magnetic constants without resorting to additional hypotheses.

This failure is not confined to the electro-magnetic theory; all optical theories labor under the same difficulty. There is, however, one distinct and important advantage which the electro-magnetic theory has over other optical theories. It is the definiteness of its premises. For whereas the dynamical theories of light must start with the material attributes, density and elasticity, of an unknown substance, the ether, which possibly may not resemble ordinary matter and whose material attributes, if it has them at all, seem to be entirely outside of the possibility of direct experimental test, the electro-magnetic theory endows the same substance with no other attributes excepting those contained in Maxwell's first and second law of electro-dynamics and the law of flux, and it proves the actual existence of these attributes by direct experiment covering a large region of electro-magnetic phenomena, the region of the Hertzian waves. It is, therefore, perfectly sure of its ground in one extensive region of ether manifestations.

The limitations of the Maxwellian hypotheses are also very clear from the character of the experiments which guided the formation of these hypotheses. These experiments were performed with constant or very slowly-varying forces, hence the obvious probability that Maxwell's hypotheses need a modification in the case of electro-magnetic phenomena involving the operation of rapidly-varying forces; in other words, Maxwell's electro-magnetic theory must be extended so as to include the periodicity of its fundamental vectors before it can become a satisfactory optical theory.

The first and the second law of Maxwell's electro-dynamics in the form stated above contain no mention of the physical constants of the field. If our experience with optical phenomena can be relied upon to guide us in our attempt to extend the electro-magnetic theory, then we should naturally infer that the element of periodicity can enter into Maxwell's electro-magnetic theory through the physical constants, only, of the field, and therefore through the law of flux and not through the two fundamental laws, which, in my opinion, do not contain these constants explicitly. It is, therefore, the law of flux which in all probability must be extended so as to include the periodicity of the impressed forces, if Maxwell's electro-magnetic theory is to become a theory of light. A careful analysis of the several satisfactory electro-magnetic theories of dispersion and absorption and of the magnetic rotation of the plane of polarization supports this conclusion. I refer in particular to the following investigations and reserve a more detailed discussion of this matter for a future occasion:

Von Helmholtz, Wied. Ann. xlviii, pp. 389 and 723, 1893. Goldhammer, Wied. Ann. xlvii, pp. 93 and 265; xlvi, p. 71, 1892. Drude, Wied. Ann. xlvi, p. 353, 1892; xlviii, p. 122 and p. 356; xlix, p. 960, 1893. Ebert, Wied. Ann. xlviii, p. 1, 1893; li, p. 268, 1894.

3. *Extension of the law of flux.*

The question arises now, can we without resorting to any hypotheses not already contained in Maxwell's electro-magnetic theory find a more general and satisfactory form for the law of flux? This question can be answered in the affirmative, provided that by a law of flux we mean a quantitative relation between the electric force and the electric flux on the one hand and the magnetic force and magnetic flux on the other.

Consider two large conductors insulated from each other by a perfect non-conductor. Let one of them be connected to one pole of a galvanic cell and the other to the other pole. We know that as soon as these conductors are connected in the manner described an integral transient current takes place whose value depends on the capacity of the conductors. The work of the cell appears half as heat in the conducting parts and half as potential energy in the dielectric. If Q be the integral current and E the electromotive force of the cell, then

$$EQ = \text{total work of the cell.}$$
$$\tfrac{1}{2}EQ = \text{potential energy stored in the dielectric.}$$

Let X, Y, Z be the components of the intensity of the electromotive reaction at any point of the dielectric and at any moment during the flow of the integral current.

Let $\frac{df}{dt}, \frac{dg}{dt}, \frac{dh}{dt}$ be the components of the intensity of the electric displacement current at the same point and time, then

$$\int_0^t dt \int \left(X\frac{df}{dt} + Y\frac{dg}{dt} + Z\frac{dh}{dt} \right) d\tau = \tfrac{1}{2}EQ$$

where $d\tau$ is an elementary volume of the dielectric. The integration extends over the whole dielectric field and over the interval of time from the start to the completion of the integral current. It is evidently immaterial how X . . . . and f . . . . vary during the flow of the integral current; the final resultant potential energy in the dielectric is, according to experiment, always the same and equal to $\tfrac{1}{2}EQ$. According to their physical character X . . . f . . . are, of course, finite, continuous, and singly-valued functions of the time; we can apply, therefore, to the indicated integration all the rules of calculus which are applicable to such functions. Integrating by parts we obtain

$$\int(X_1f_1+Y_1g_1+Z_1h_1)d\tau-\int_0^t dt\int\left[f\frac{dX}{dt}+g\frac{dY}{dt}+h\frac{dZ}{dt}\right]d\tau=\tfrac{1}{2}EQ$$

where the suffix 1 denotes that the value at the moment of cessation of the integral current is to be taken.

Now it can be easily shown that the first integral equals EQ. Hence

$$\int_0^t dt\int\left\{\left(X\frac{df}{dt}-f\frac{dX}{dt}\right)+\left(Y\frac{dg}{dt}-g\frac{dY}{dt}\right)+\left(Z\frac{dh}{dt}-h\frac{dZ}{dt}\right)\right\}d\tau=0$$

Every functional relation between $X\ldots f\ldots$ which will satisfy this equation will be a permissible form of the law of flux.

For instance,

$$f=\frac{K}{4\pi}X,\ g=\frac{K}{4\pi}Y,\ h=\frac{K}{4\pi}Z$$

is one set of relations which will satisfy this equation.

Here is another set

$$f=\frac{1}{4\pi}(K_{11}X+K_{12}Y+K_{13}Z)$$

$$g=\frac{1}{4\pi}(K_{21}X+K_{22}Y+K_{23}Z)$$

$$h=\frac{1}{4\pi}(K_{31}X+K_{32}Y+K_{33}Z)$$

where $K_{12}=K_{21},\ K_{13}=K_{31},\ K_{23}=K_{32}$

By a suitable selection of coördinate axes this set can be written

$$f=\frac{K_1}{4\pi}X$$

$$g=\frac{K_2}{4\pi}Y$$

$$h=\frac{K_3}{4\pi}Z$$

A third set of relations is obtainable by supposing f, g, h to vary harmonically, since in this case

$$\frac{d^{2n}f}{dt^{2n}}=(-1)^n p^{2n}f,\ \text{etc.}$$

where $p=2\pi\times$ frequency, we obtain

$$f = \left\{ \frac{4\pi}{\frac{4\pi}{K_1} - p^2 A} \right\} X$$

$$g = \left\{ \frac{4\pi}{\frac{4\pi}{K_2} - p^2 B} \right\} Y$$

$$h = \left\{ \frac{4\pi}{\frac{4\pi}{K_3} - p^2 C} \right\} Z$$

as another possible set of relations.

Here
$$A = a_1 - a_2 p^2 + a_4 p^4 - \dots.$$
$$B = b_1 - b_2 p^2 + b_4 p^4 - \dots.$$
$$C = c_1 - c_2 p^2 + c_4 p^4 - \dots.$$

In this permissible form of the law of flux the constant which corresponds to specific inductive capacity is a function of the periodicity. A large variety of other relations can easily be found all of which will satisfy the integral equation.

If the dielectric is absorptive then the value of the integral does not vanish but equals the heat developed in the dielectric. Hence in general

$$\text{(C)} \int_0^t dt \int \left\{ \left(X\frac{df}{dt} - f\frac{dX}{dt} \right) + \left(Y\frac{dg}{dt} - g\frac{dY}{dt} \right) + \left(Z\frac{dh}{dt} - h\frac{dZ}{dt} \right) \right\} d\tau \geqq 0$$

contains all the possible forms of the law of flux relating to the electrical force and the electric flux, which are consistent with Maxwell's hypotheses and with the quadratic form of electropotential energy.

A similar relation between the magnetic force and the magnetic flux can be obtained. Consider a loop of wire in which a galvanic cell sets up a current. From the moment of closing the circuit up to the time when the current becomes constant magnetic energy is stored up in the medium. The final amount of magnetic energy will be in the well known notation

$$W = \tfrac{1}{2} LC^2$$

Let α, β, γ be the components of the intensity of the magnetomotive reaction at any point of the field and at any moment during the flow of the magnetic integral current.

Let $\frac{da}{dt}, \frac{db}{dt}, \frac{dc}{dt}$ be the components of the intensity of the magnetic current at the same point and the same moment, then

$$\int_0^t dt \int \left(\alpha\frac{da}{dt} + \beta\frac{db}{dt} + \gamma\frac{dc}{dt} \right) d\tau = \tfrac{1}{2} LC^2$$

Integrating by parts we obtain

$$\int(\alpha_1 a_1+\beta_1 b_1+\gamma_1 c_1)d\tau-\int_0^t dt\int\left(a\frac{d\alpha}{dt}+b\frac{d\beta}{dt}+c\frac{d\gamma}{dt}\right)d\tau=\tfrac{1}{2}\mathrm{LC}^2$$

The first integral refers to the state of the field when the electric current has become constant and therefore the magnetic integral current has ceased to flow. The value of this integral can be easily shown to be LC^2 and we have, therefore,

$$\int_0^t dt\int\left\{\left(\alpha\frac{da}{dt}-a\frac{d\alpha}{dt}\right)+\left(\beta\frac{db}{dt}-b\frac{d\beta}{dt}\right)+\left(\gamma\frac{dc}{dt}-c\frac{d\gamma}{dt}\right)\right\}d\tau=0$$

The same relations can be deduced from this integral as from (C). If the dielectric of the field is absorptive then the last integral equals the heat developed in the field. We can put, therefore

$$\text{(D)}\int_0^t dt\int\left\{\left(\alpha\frac{da}{dt}-a\frac{d\alpha}{dt}\right)+\left(\qquad\right)+\left(\qquad\right)\right\}d\tau\geqq 0$$

as the most general relation between the magnetic force and the magnetic flux.

Equations (C) and (D) are the consequence of the following premises:—First, the hypotheses by means of which we pass from the two experimental laws of electro-magnetic and magneto-electric induction to Maxwell's two fundamental laws are correct. Second, it is an experimental fact that the electric energy of the field is proportional to the square of the integral electric and the magnetic energy is proportional to the square of the integral magnetic current which has passed across the field. Third, the principle of conservation of energy is applicable to the processes just described.

Since these equations (C) and (D) contain as a special case not only that form of the law of flux which Maxwell employed in the second part of his electro-magnetic theory, but also a large variety of other forms of that law, in which the physical constants which determine the propagation of an electro-magnetic disturbance appear as functions of the periodicity of the impressed forces, and since forms of the law of flux of the same type as these latter forms were assumed hypothetically in most of the recent developments of the electro-magnetic theory of light because they lead to a tolerably satisfactory explanation of the phenomena of dispersion, absorption, rotation of the plane of polarization,—it follows that we can accept these two equations as the most general expression for the law of electro-magnetic flux.

Summary.

The main points of this essay can be summed up as follows:

Maxwell's electro-magnetic theory consists of two *distinct* parts. The essential elements of the first part are the law of induced electromotive force and the law of induced magneto-motive force. These two fundamental laws are an extension of the two experimental laws of magneto-electric and of electro-magnetic induction, which extension consists in adding to what is already contained in these experimental laws the hypotheses that the magnetic and the electric flux in dielectrics are not a mere mathematical fiction, as the older electro-magnetic theories supposed, but an actually existing state in the dielectric whose rate of variation through any imaginary surface in the dielectric is proportional to the reacting electro-motive, respectively magneto-motive force, induced around the circuit of the boundary line of this surface. The value of the reacting force is independent of the nature of the substance through which the circuit passes. Hence the physical constants of the electro-magnetic field (that is specific inductive capacity, etc.) do not appear in the first part of Maxwell's electro-magnetic theory.

The second part of this theory takes account of these constants by adding to the two fundamental laws just mentioned the ordinary form of law of flux. The first two laws are a cross-connection between the varying flux of one type and the reacting force of the other type; the law of flux, on the other hand, is a direct connection between the force and the flux of its own type. Combining the two fundamental laws with the law of flux, we obtain the fundamental differential equations of the second part of Maxwell's electro-magnetic theory. These equations are a mathematical statement of the laws of propagation of an electro-magnetic disturbance through various media, exhibiting the remarkable fact that, on the one hand, the velocity of propagation of an electro-magnetic disturbance is approximately and the wave-form of propagation is exactly the same as in the case of light; on the other hand, however, the velocity of propagation of an electro magnetic wave through a non-absorptive dielectric is independent of its periodicity, a fact which does not hold true in the case of light.* Hence although both parts of the Maxwellian theory agree well with experimental facts which were brought to light by the labors of Hertz and of other physicists who extended the Hertzian methods of investigation, yet this theory falls short of being a satisfactory theory of light, because it represents the propagation of electro-magnetic waves as inde-

* An apparent exception to this is the propagation of light through the most perfect of all dielectrics, namely, the ether in a perfect vacuum. The velocity of propagation of light through this dielectric is, within the narrow limits of the visible spectrum, independent of the wave length. The question, however, whether all wave-lengths travel through pure ether with the same velocity has not yet been answered definitely. The statement, therefore, that the velocity of Hertzian waves in ether is the same as that of light is somewhat indefinite.

pendent of the periodicity of these waves, and therefore fails to account for some of the most important optical phenomena like dispersion, absorption, etc. These phenomena we know to be due to the dependence of the optical constants of a substance on the periodicity of waves; hence, if Maxwell's electromagnetic theory is to become a satisfactory theory of light, it must be extended so as to represent the electro-magnetic constants of a substance as functions of the periodicity of the impressed electro-magnetic forces. This extension of the Maxwellian theory will affect, therefore, the law of flux only, and not the first part of this theory, since this part does not and should not contain any reference whatever to the nature of the medium of which the electro-magnetic field is composed.

This opinion is confirmed by the recent developments of the electro-magnetic theory of light, since in these the law of flux and not the two fundamental laws of Maxwell's theory appear in a modified form. Besides, it is obvious that Maxwell's form of the law of flux holds true for constant and slowly varying forces only. In every other case its application is only tentative. The second part of the Maxwellian theory should, therefore, be considered as a tentative and therefore approximate description of the laws of propagation of electro-magnetic disturbances. A more accurate description of these laws must proceed from a more general form of the law of flux, a form which will hold true for forces of all periodicities. That such a form of the law of flux actually exists is rendered highly probable by considering the most general relation between the electric and magnetic forces and their fluxes which will satisfy Maxwell's fundamental hypotheses, and also the observed fact that the electric, respectively magnetic, energy of the medium through which an integral electric, respectively magnetic, current *has passed* is a quadratic function of that integral current. This general relation is expressed by the forms (C) and (D) deduced above.

Maxwell's provisional form of the law of flux is a special case of this general relation. This general relation suggests also that many other forms of the law of flux are permissible in which those physical constants which determine the propagation of an electro-magnetic disturbance appear as functions of the periodicity of the disturbance. But these constants are not specific inductive capacity, magnetic permeabilities and resistivity of the medium.

It still remains to be shown that according to Maxwell's electro-magnetic theory a law of flux containing explicitly the periodicity of the impressed forces is not only admissible but also necessary. When that is shown, then this theory will have fulfilled one of the most essential conditions which every satisfactory theory of light must fulfill.

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