ART. XLIII.—*An elementary proof of the Earth's Rigidity;* by GEO. F. BECKER.

Definition of rigidity.—The term rigidity, as used in the theory of elastic bodies and in the theory of the earth, denotes any degree of resistance to change of shape. It must be confessed that the selection of this word to express this quality is not a fortunate one. In ordinary mechanics, as in ordinary life, a rigid body means one which is incapable of change of shape. Such a body would be called by elasticians *infinitely* rigid, and they attribute to every substance which shows elasticity of shape, however small, or in other words to every solid substance, the property of rigidity. Thus both india-rubber and steel are in their language rigid bodies, the difference in this respect being that the "modulus of rigidity" of steel is much greater than that of rubber. From the ambiguity of the term rigid, some geologists who were unfamiliar with his writings have been led to suppose that Sir William Thomson maintained that the earth is incapable of deformation. So far is this from the truth that Thomson was the first physicist who investigated the tidal problem on the hypothesis that the earth undergoes deformation by the attraction of the moon or sun. It is to be understood, then, that the title of this paper may be stated less concisely, but without change of meaning, as "an elementary proof of the earth's resistance to change of shape." It will not be found that the term rigidity, once understood in its special sense, leads to confusion.

Difficulty of Thomson's Problem.—No geologist would think of denying that the question of the earth's solidity is the most important in physical geology, but as it has been treated by physicists, the subject is one of peculiar difficulty. Sir William Thomson has shown that the earth exhibits to the attractive forces of the sun and moon a resistance so great as to imply that it is solid throughout. To reach this conclusion it is necessary to determine separately the resistance to deformation which a solid sphere presents because of the mutual attraction of its parts, and the resistance due to the elastic forces which are called in play when deformation occurs.

The resistance due to gravitation is thoroughly well understood and this portion of the subject is not hard to master. On the other hand, the determination of the elastic resistance is very difficult. The general theory of the statics of elastic bodies is highly complex, and physicists are by no means agreed as to whether the general equations involve 21 or only 15 independent constants.* Both schools accept as the empirical basis of calculation Hooke's law, which experiments show to be only approximately true for small pressures, while for high pressures it is morally certain that the so called constants assume other values, and no physicist professes to be able to obtain results which are strictly accurate excepting for infinitesimal deformations.—On the basis of the general elastic theory, Lamé investigated the equilibrium of an elastic sphere by the method of spherical harmonics. There seems to be no doubt that this investigation is a masterpiece of genius. Thomson's inquiry is based upon Lamé's. He has considerably simplified the discussion, but without rendering it either short or easy; and he has applied it to the case of a homogeneous, isotropic sphere, of the size and mean density of the earth, without mutual gravitation of its parts, but subject to the attraction of a distant external body such as the sun or the moon. That even this application of Lamé's problem is laborious may be inferred from the fact that Thomson himself refers to the "tedious algebraic reductions" which he omits.

The result, so far as the earth is concerned, is an extremely simple formula for the ellipticity of an elastic sphere.

When this formula is once reached, Thomson's argument for the great effective rigidity of the earth presents no difficulty which may not be made intelligible to any geologist. It is not strange, however, that there has been some reluctance in geo-

* B. de Saint-Venant, probably the greatest elastician up to this time (he died in 1886), followed Poisson in maintaining that eolotropic elasticity involves only 15 moduluses and that isotropic elasticity involves but 1 constant. On the other hand, Green, Stokes, Thomson, and others, starting from a different view of the molecular constitution of matter, argue that 21 and 2 moduluses respectively are requisite. Lamé adopted the molecular theory which leads to uniconstant isotropy, but expresses his results by biconstant formulas. These are certainly the more convenient, for they can at once be reduced to uniconstant forms. See Todhunter's History of Elasticity.

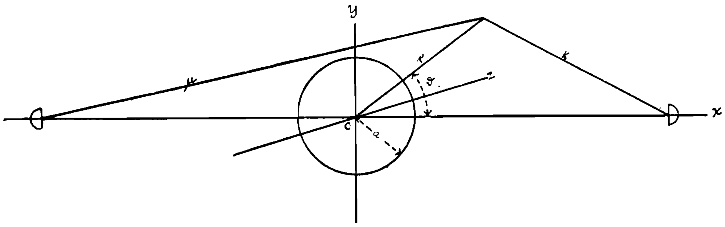
logical circles to admit the necessity of a result the demonstration of which scarcely anyone—perhaps no one—generally classed as a geologist can follow critically.

Possibility of a simplification.—It occurred to me that, since even the result of the Lamé-Thomson investigation is only an approximation for real (or finite) strain there might be some simpler way of reaching another approximation which, though less accurate, would be sufficiently close for a demonstration of the rigidity of the earth. Thomson's result is that the ellipticity of the elastic sphere, under the limitations mentioned above, is $\frac{5}{19}$ of a certain constant. If one could show that this ellipticity were, at all events, greater than $\frac{1}{4}$ of this same constant, it will appear by an examination of Thomson's argument that substantially the same results would follow.

I have succeeded in proving this inequality by a method so simple that no mathematics beyond plane trigonometry is needful to follow it. Besides presenting this proof in the most elementary possible manner to my colleagues, I shall take the occasion to follow out Thomson's argument in the way which seems best adapted to geological readers, for it appears to me certain that on some points he has been very generally misunderstood.

General character of the earth's elastic strain.—The problem of the deformation of an elastic sphere by the attraction of an external body like the moon is closely analogous to the simplest tidal problem. Suppose that the earth were always to present the same face to the moon, just as the moon presents (nearly) the same face to the earth. Then the waters would be drawn upward toward the moon to a definite ellipsoidal surface capable of accurate computation. This ellipsoid would have three unequal axes, of which the longest would be directed toward the moon, while the polar axis would be the shortest. The difference between the polar axis and the shorter of the equatorial axes would be due to the rotation of the earth-moon system, and but for this rotation the two bodies would fall together. This rotation, however, produces an elliptical flattening of the globe, while the attraction of the moon produces a distinct but superimposed elongation of the globe. It is desirable to separate these two effects and this can be done by the following device. The earth may be supposed at rest in space and subjected to the action of two bodies fixed at equal distances from it in opposite directions. If each of these bodies has half the mass of the moon, and if the distance of each of them from the earth's center is equal to the moon's real mean distance, then the deformation will be sensibly the same as that which the real moon would produce irrespective of the effects of rotation. It is the tide found on this hypothesis which is known as the "equilibrium tide."* In the elastic problem, also the globe is supposed subjected to the attraction of two half-moons, but the resistance to deformation is supposed due solely to elastic forces instead of to gravitation.

Form of the stress.—Let the earth's center of inertia be taken as the center of coördinates; let r be the distance of any



point in space from this origin and let ϑ be the angle which r makes with the x axis. Let also μ and $\varkappa$ be the distances of the point from the two portions of the moon, and let M be the mass of the moon and D its mean distance. Then, by the law of gravitation, the forces acting on the point are

$$\frac{M}{2\varkappa^2} \text{ and } -\frac{M}{2\mu^2}$$

and here

$$\varkappa^2 = D^2\left(1 + \frac{r^2}{D^2} - \frac{2r}{D}\cos\vartheta\right);\ \mu^2 = D^2\left(1 + \frac{r^2}{D^2} + \frac{2r}{D}\cos\vartheta\right).$$

These forces can of course be resolved in any direction by simple projections. The resulting formulas can also be simplified without substantial loss of accuracy by neglecting powers of r/D higher than the first. The distance of the moon is 60·3 times the radius of the earth, and the highest value of r^2/D^2 which can occur in this discussion is therefore only 1/3,636.

If the forces are resolved into F_r, the component coinciding in direction with the radius vector, and F_t the component at right angles to the radius it will be found that

$$F_r = \frac{Mr}{D^3}(3\cos^2\vartheta - 1) \text{ and } F_t = \frac{Mr}{D^3}\, 3\sin\vartheta\cos\vartheta.$$

Or if the forces are resolved into components X, Y, Z in the directions of the axes (reckoned as usual per unit volume)

* This system of "moon and anti-moon" is that employed by Laplace, Thomson and others. Prof. G. H. Darwin has given a far more elegant method of dealing with the equilibrium tide (Encycl. Brit., Article Tides, 1888); but though his treatment is very simple, it is less adapted to a paper like this, from which it seems best to exclude the potential.

$$X=\frac{2Mx}{D^3};\quad Y=\frac{My}{D^3};\quad Z=\frac{Mz}{D^3}.$$

These, or equivalent forms, are those used in the theory of the tides and are amply accurate for most problems connected with the subject.

The total values of these forces acting on the earth are easily found. Consider first the hemisphere to the right of the yz plane. The force acting on each particle is simply proportional to its distance from the dividing plane, or to x. Now it is well known that if ξ is the distance of the center of inertia of a mass from a given plane, and if m is the mass of an elementary particle $\xi\Sigma m=\Sigma\ (mx)$. Hence the total force will be proportional to the mass of the hemisphere into the distance of its center of inertia from the origin. For the hemisphere $\xi=3a/8$ where a is the radius, and if w is the mean density, the whole force acting on the basal plane is say

$$\pi a^2 X=\frac{2M}{D^3}\Sigma(mx)=\frac{2M}{D^3}\cdot\frac{3}{8}a\cdot\frac{2}{3}\pi a^3 w=\frac{2M}{D^3}\frac{\pi a^4 w}{4}. \qquad (1)$$

In precisely the same way one finds for the sum of the forces affecting the hemisphere which lies above the xz plane

$$\pi a^2 Y=\frac{M}{D^3}\frac{\pi a^4 w}{4}$$

and the symmetry of the figure shows that $Z=Y$.

The deformation of an elastic sphere of the size and mean density of the earth by a mass so small as the moon (1/83 of the mass of the earth) and at such a distance from it, is of course very minute. The most probable deformation in fact amounts to an ellipticity of about 1/10,000,000. It may be inferred therefore that the deforming stress on the plane passing through the earth's center of inertia at right angles to the direction of the moon is not very far from uniform. Were this the case the earth would be homogeneously strained, so that each elementary cube would suffer the same distortion.

Simple strain spheroid.—Suppose a cube of isotropic, homo geneous, incompressible matter (like india rubber) subjected to a uniformly distributed tensile stress in the direction of the x axis. Then by Hooke's law the elongation per unit length will be proportional to the force per unit area, say X; and if n is the constant called the modulus of rigidity, this elongation is $X/3n$. If the mass is but little strained and the volume is constant, this elongation is accompanied by a linear lateral contraction in each direction which is equal to half the elongation.*

* If the elongation of the unit cube is a and the linear lateral contraction is β, while the volume remains constant, $1=(1+a)(1-\beta)^2$; and if a^2 and $a\beta$ are so small as to be negligible, this reduces to $a=2\beta$.

Suppose that the cube were also subjected to a compressive force Y acting at right angles to X and to a compressive force Z at right angles to each of the others. Then if the height of the cube is $2a$ and if α, β, γ are the displacements on the three axes, the cube will be elongated in the direction of x by

$$\alpha = \frac{2aX}{3n} + \frac{2aY}{6n} + \frac{2aZ}{6n}$$

and it will be contracted in the direction of y by

$$\beta = \frac{2aX}{6n} + \frac{2aY}{3n} - \frac{2aZ}{6n}$$

and γ will be symmetrical with β.

In a special case let $2Z=2Y=X$. Then

$$\alpha = \frac{2aX}{2n}; \quad \beta=\gamma=\frac{2aX}{4n}.$$

Now this is precisely the effect which would be produced by a simple tensile force, P in the direction of x, unaccompanied by lateral forces, if

$$P = \frac{3}{2} X. \tag{2}$$

Consider a sphere of radius a inscribed in the cube before the application of the force P. After the mass was strained the sphere would have become an ellipsoid the major semi-axis of which would be

$$a+\frac{\alpha}{2} = a\left(1+\frac{X}{2n}\right) = a\left(1+\frac{P}{3n}\right)$$

and the minor semi-axes would be each

$$a-\frac{\beta}{2} = a\left(1-\frac{X}{4n}\right) = a\left(1-\frac{P}{6n}\right)$$

The ellipticity of this ellipsoid would be

$$e = 1 - a\frac{\left(1-\frac{P}{6n}\right)}{a\left(1+\frac{P}{3n}\right)} = \frac{P}{2n} \cdot \frac{1}{1+\frac{P}{3n}}$$

Here P is by supposition a force so small that its square can be neglected and thus

$$(1+P/3n)^{-1} = 1 - P/3n$$

and hence also

$$e = \frac{P}{2n} \tag{3}$$

First approximation to the earth's bodily tide.—On the hypothesis that the earth is homogeneously strained, I have only to find P of the last formula in terms of the forces applied to the earth. This has really been done in a preceding paragraph. Equations (1), (2) and (3) give

$$P = \frac{3}{2}X = \frac{3}{2} \cdot \frac{2M}{D^3} \cdot \frac{a^2 w}{4} = \frac{1}{2}\tau a^2 w$$

where $\tau = 3M/2D^3$ is introduced for brevity; and if e_s is the ellipticity of the earth regarded as a simple strain ellipsoid

$$e_s = \frac{5}{20n}\tau a^2 w.$$

Character of the approximation.—No use can be made of the value e_s unless it can be shown that it is less than the true value. It has been found on the hypothesis that the stress is uniformly distributed and that the mass is incompressible, and neither supposition is strictly correct.

If one considers the true distribution of stress, it is almost self-evident that the applied force will be greater near the major axis than near the periphery of the yz section. But this can easily be proved. Suppose a slender cylinder of the elastic spheroid close to its major axis, separated from the surrounding mass as if by a diamond drill. Then the force acting upon this cylinder will be its mass into the distance of its center of inertia from the origin. But this distance is $a/2$ while the center of inertia of the hemisphere is only $3a/8$ from the origin. The cylinder will therefore be much more elongated than it would be if the sphere were homogeneously strained.

When this axial cylinder is connected with the surrounding mass, it will be held back to some extent and cannot be so much elongated as if it were free. But the fact that, when free, it is elongated by more than the average amount per unit length, proves that even when under constraint there is a tendency to greater elongation, which must be operative to some extent.

If the two lateral compressions due to the forces Z and Y are considered, it is clear that these strains will also be more intense near their respective axes than is assumed on the hypothesis of homogeneous strain. Thus the approximation underestimates the longitudinal extension and underestimates the lateral contraction; in short, it underestimates the ellipticity of an incompressible mass.

The famous result of Sir William Thomson for the case of incompressibility is, say,

$$e_r = \frac{5}{19n}\tau a^2 w = 1{\cdot}05 e_s$$

Influence of compressibility.—It will be well to consider first the effect of compressibility on the homogeneously strained spheroid already discussed. If k is the coefficient of compressibility (becoming ∞ when the mass is incompressible) the elongation of a cube of height $2a$ becomes

$$2a\,\mathrm{X}\left(\frac{1}{3n}+\frac{1}{9k}\right)$$

and the accompanying lateral contraction is

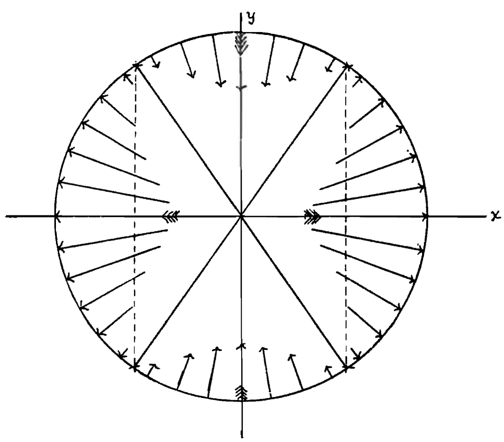
$$2a\,\mathrm{X}\left(\frac{1}{6n}-\frac{1}{9k}\right).^*$$

The elongations and contractions for Y and Z are of course of the same form. If these values are introduced into the expressions for α, β, γ, it will be found that the terms in k cancel, and that the ellipticity is exactly the same as for the case of incompressibility.

To find the effect of compressibility upon the sphere when the distribution of strains is the real one, it is more convenient to deal with the polar components of the forces or

$$\mathrm{F}_r=\frac{\mathrm{M}r}{\mathrm{D}^3}\left(3\cos^2\vartheta-1\right);\ \mathrm{F}_t=\frac{\mathrm{M}r}{\mathrm{D}^3}3\sin\vartheta\cos\vartheta.$$

The latter or tangential component will have no effect upon the volume. The forces of this form will form couples tending to increase the angle which each diameter originally made with



the yz plane. Compression or dilatation will be produced by the component which is directed toward or from the center, or

* For the very simple method by which these formulas are established, see Thomson and Tait, Natural Philosophy, § 683, or Thomson's article on Elasticity, in the Encycl. Brit.

by F_r. This force vanishes when $3\cos^2\vartheta=1$ or when $\vartheta=54° 44'$, nearly. The lines of no compressive force form a cone of two sheets, the axis of which coincides with the line connecting the two halves of the moon. Within this cone the forces F_r are directed from the center and tend to produce dilatation of the mass, and F_r is a maximum on the axis where it has the value $2\frac{Mr}{D^3}$. Between the sheets of the cone is an annular mass bounded on the surface by a zone. Within this mass the forces are directed inward, and tend to produce compression. The compressive force is maximum when $\vartheta=90$ and thus has the value Mr/D^3.

This distribution is illustrated in the diagram on page 343, where an arbitrary length is assumed as equal to the value of the force on the x-axis and the other forces are exhibited in the true proportion to it.*

When strains are small, they may be supposed to take place successively. Thus one may arrive at the distortion of a compressible sphere by supposing it first deformed as if it were incompressible, and that it then becomes compressible. If this course is adopted in the present case, the incompressible sphere would be distorted to an ellipsoid of the same volume, in which the compressive stresses exhibited in the diagram would exist, but would be inoperative. If compressibility now supervenes, it is clear that dilatation will take place along the major axis and compression at right angles to it.

General result for elastic spheroid.—Hence compressibility must increase the ellipticity which an elastic sphere assumes under a tide-generating stress. Hence also the ellipticity is always greater than $\tau wa^2/4n$, and any conclusions which can be

* It is a noteworthy fact that *all* ellipsoids of infinitesimal ellipticity and constant volume intersect the surface of the unstrained sphere at the circles of no force. One way of proving this is as follows: Let the semi-axes of the ellipse be $a(1+\delta)$ and $a(1-\delta/2)$ where δ is so small that its square can be neglected. Then, the volume being constant,

$$a^3 = a(1+\delta)\,.\,a^2(1-\delta/2)^2$$

and the equation of the ellipse is

$$\frac{x^2}{(1+\delta)^2}+\frac{y^2}{(1-\delta/2)^2} = a^2 = x^2(1-2\delta)+y^2(1+\delta)$$

when this ellipse cuts the circle, $a^2=x^2+y^2$, and

$$2x^2=y^2 \text{ or } \tan^2\vartheta=2 \text{ and } \cos^2\vartheta=\frac{1}{3}$$

irrespective of the spherical value of δ.

The equipotential surfaces are hyperboloids of revolution asymptotic to the cone mentioned in the text, and the equation of the potential is simply

$$\int F_r\,dr = V = \frac{Mr^2}{2D^3}(3\cos^2\vartheta-1)$$

drawn as to the rigidity of the earth by taking as a minimum value of the distortion

$$e_s = \tfrac{1}{4}\,\frac{\tau w a^2}{n}$$

are perfectly safe.

If $3k = 5n$, as is the case for the theoretical, uniconstant, isotropic solid, and at least approximately also for glass and iron, the result reached by the beautiful, but laborious, method of Lamé and Thomson is, say

$$e_c = \frac{5}{19}\,\frac{\tau w a^2}{n}\left(1 + \tfrac{2}{55}\right) = 1{\cdot}09\ e_s$$

so that one is very safe indeed in taking the ellipticity as at least e_s.

Ellipticity of a fluid sphere.—Having considered the elastic resistance of the earth apart from the influence which the mutual attraction of its elements produces, it is next necessary to consider the effect of this attraction apart from the rigidity, or as if the mass were a fluid. Later it will appear how the results are to be combined so as to include both elasticity of form and attraction toward the center.

It is very easy indeed to prove that if the earth were a homogeneous, incompressible fluid of mean density 5·5, and if the attraction were directed to a single, central point; the ellipticity of the equilibrium tide would be $a\tau/g$, where g is the acceleration of pure gravitation.* It is somewhat more difficult to prove that the mutual attraction of the fluid in its deformed state would augment this ellipticity by 5/2. But these are familiar propositions of analytical mechanics, which no one doubts and which geologists are safe in accepting.

The ellipticity of an incompressible homogeneous fluid earth would then be, say,

$$e_g = \frac{5}{2}\,\frac{a\tau}{g} = 162 \times 10^4 \tau.$$

* The forces acting on a fluid sphere by the equilibrium theory are $\frac{M}{2\mu^2}$, $\frac{M}{2\kappa^2}$, and gravitation which is expressed by $\frac{E}{a^2}$ where E is the earth's mass. This leads to a value of the major semi-axis expressed by $r = a\left(1 + \frac{Ma^3}{ED^3}\right)$. It is also readily proved that for any ellipsoid closely approaching the sphere, the major semi-axis is $a\left(1 + \frac{2}{3}e\right)$. Hence $e = \frac{3Ma^3}{2ED^3}$. (In Nat. Phil., § 804 this ellipticity is erroneously stated at twice this quantity). The ordinary meaning of g is the resultant of gravitation and centrifugal force: but on the equilibrium theory there is no rotation of the globe, and g is then equal to E/a^2. Substituting this and the value of τ, the equation becomes $e = a\tau/g$.

If the fluid supposed at first incompressible were to become compressible, it is clear that a would decrease and that g, being inversely proportional to the square of the distance from the earth's center, would increase. Thus the resistance to deformation by the attraction of the moon would increase and the deformation measured by the ellipticity would decrease. It can also be shown in detail that if the increase in density took place in any manner capable of approximate expression by Laplace's law (that the increase of the square of the density is proportional to the increase of the pressure) compressibility would decrease the ellipticity; but even infinite compressibility would only reduce the ellipticity by a moderate fraction of its value for the case of incompressibility.* Beyond a knowledge of the sense in which compressibility would affect the ellipticity of the fluid sphere it is really not worth while to inquire, for it is not at all probable that the so-called constants of elasticity, n and k, would exhibit invariability if we could experiment with pressures of hundreds of thousands of pounds to the square inch; indeed there is good evidence of great changes in rigidity even at pressures which have been experimentally applied.

Comparison of results.—In the following table I have collected the various ellipticities mentioned above with numerical expressions for spheres of the mean density of the earth, but with the rigidity of glass, brass and steel. Thomson gave figures for glass and steel, and I have added those for brass. I

* Centrifugal force is equivalent to a simple repulsion from a line, and if ω is the angular velocity, the measure of this force is $\omega^2 r$. Tide-generating force is equivalent to repulsion from a plane as appears from the manner in which it was deduced above and its measure is $2\tau x$. Hence the formulas for the two cases are convertible by the substitution of one of these forces for the other, bearing in mind that centrifugal spheroids are oblate and that tidal ellipsoids are prolate. See Nat. Phil., § 834, page 432.—If E is the mass of the earth, and if f is the ratio which the mean density of a sphere bears to its surface density, the ellipticity of a rotating spheroid in which there is a small increase in density from the surface downward, may be expressed by

$$e = \frac{5}{2}\frac{\omega^2 a^3}{E}\cdot\frac{1}{2(1+4(f-1))}$$

(Cf. Nat. Phil., §§ 824′ and 800). In the case of the equilibrium tide, real and apparent gravitation coincide and therefore $g = E/a^2$. Introducing also the value of τ this becomes

$$e = \frac{5}{2}\cdot\frac{\tau a}{g}\cdot\frac{1}{1+4(f-1)}$$

Similarly § 824′(2) gives for the case in which the surface density is infinitesimal as compared with the density at the center

$$e = \frac{6}{\pi^2}\cdot\frac{5}{2}\frac{a\tau\omega}{g} = \text{nearly } \frac{6}{10}\cdot\frac{5}{2}\cdot\frac{a\tau\omega}{g}$$

For the reasons stated in the text these formulas are of no use excepting to show that progressive increase of density decreases the ellipticity of a fluid sphere.

may note that, judging from the value of "Young's modulus," cast iron and phosphorus bronze have a rigidity nearly the same as that of brass. The rigidity of brass is from a determination by Everett. In the table n/g is the value of n in grams weight per square centimeter; e_s is the ellipticity of a homogeneously strained sphere, treated in this paper as affording a rough approximation; e_r is the ellipticity as given by Thomson's formula for the case of incompressibility; e_c is the value given by his formula for the case of compressibility if $3k = 5n$, or if isotropy involve but a single constant. Finally S and R, which will be of use presently, are given by

$$S = \frac{e_s}{e_g + e_s} \quad \text{and} \quad R = \frac{e_r}{e_g + e_r}$$

Ellipticities of Elastic, Non-Gravitating Spheroids.

	n/g	e_s	e_r	e_c	S	R
Glass	244×10^6	$235 \times 10^4\tau$	$248 \times 10^4\tau$	$257 \times 10^4\tau$	0·591	0·606
Brass	373×10^6	$153 \times 10^4\tau$	$161 \times 10^4\tau$	$167 \times 10^4\tau$	0·485	0·498
Steel	780×10^6	$72 \times 10^4\tau$	$77 \times 10^4\tau$	$80 \times 10^4\tau$	0·311	0·322

Comparing these ellipticities with that for a fluid sphere, $e_g = 162 \times 10^4\tau$, it appears that in a globe of the size of the earth, the mutual attraction of parts is much more powerful than the elasticity of figure if the rigidity is that of glass; that these two resistances to deformation are substantially equal if the rigidity is that of brass; and that gravitation is less than half as powerful as elasticity of figure if the rigidity is that of steel.*

Combination of resistances.—The ellipticity of a deformed sphere varies directly as the applied force and inversely as the resistance. Thus if W is the resistance one may write

$$\frac{1}{e} = \frac{W}{\tau}$$

In the cases of an elastic incompressible sphere and a fluid sphere this equation takes the forms

$$\frac{1}{e_r} = \frac{19n}{5a^2w} \cdot \frac{1}{\tau} = \frac{\mathfrak{r}}{\tau} \quad \text{and} \quad \frac{1}{e_g} = \frac{2g}{5a} \cdot \frac{1}{\tau} = \frac{\mathfrak{g}}{\tau}$$

where the Old English letters are introduced for brevity. In the case of a sphere in which gravitation and elasticity co-operate, $W = \mathfrak{g} + \mathfrak{r}$ and the ellipticity is expressed by

* In the first sentence of Nat. Phil., § 840, it seems to me certain that Sir William Thomson must have meant to write glass instead of steel, for he has just shown that the attraction of the moon deforms a fluid sphere much more than it does a steel sphere in which there is no mutual attraction of parts.

$$\frac{1}{e_r} = \frac{\mathfrak{g}+\mathfrak{r}}{\tau} = \frac{1}{e_g} + \frac{1}{e_r}$$

or

$$e = \frac{\tau/\mathfrak{g}}{\mathfrak{r}/\mathfrak{g}+1} = e_g \cdot \frac{e_r}{e_g+e_r} = e_g \mathrm{R}$$

where R is the fraction given in the table. If instead of taking Thomson's value for the ellipticity of the elastic spheroid, or e_r, my rough approximation, e_s, is substituted, e simply becomes e_gS. In the table R and S are given in decimals in order to show how far they differ. Excepting for this purpose the third decimal is of no value whatever, indeed the second decimal is also nearly if not quite worthless. Now for glass S=0·98 R and consequently is substantially identical with R.

Both R and S are in fact underestimated; for the earth is certainly compressible to some extent and, as has been shown above, compressibility increases e_r and decreases e_g. Hence the deformation of the solid globe as compared with a fluid one of the same mean density is distinctly understated in the above formulas or in other words the globe is really more rigid than the formulas imply.* The extent to which the rigidity of the earth is understated by the formulas however must be quite insignificant, unless indeed the earth contains ellipsoidal shells exhibiting a higher degree of compressibility than is shown by ordinary solids and liquids.†

Whether R or S is used in the formulas one may thus safely state that, if the earth had the same rigidity as glass, it would yield to the deforming influence of the moon and sun 3/5 as much as if it were fluid and that if it possessed the rigidity of brass it would yield 1/2 as much as if it were fluid. Even in the case of steel S=0·97 R and the difference between R and S is far within the limits of error of any knowledge which we can ever hope to reach as to the actual rigidity of the earth. Thus for steel R or S may be taken at about 1/3, and all these values, which follow from the investigations of Lamé and Thomson are also deducible from the inequality $e > \frac{1}{4}a^2\tau w$ itself deduced by the most elementary methods.

* The Rev. Osmond Fisher has attempted to reconcile the hypothesis of a fluid substratum with the effective rigidity of the earth (Physics of the Earth's Crust, 2d edition, chapter v). If I understand him correctly, he misunderstands Henry's law on which his argument is based. He seems to suppose that the amount of gas which a fluid will dissolve is proportional to the sum of the pressures of the gas and the fluid. Henry's law shows the proportionality between the pressure of the gas alone and the amount of gas which the fluid will dissolve. Water from great oceanic depths contains no more gas in solution than that from the surface.

† Thomson after calculating the effect of the compressibility of the uniconstant solid on the ellipticity of an elastic spheroid, says "we see that the compressibility of the elastic solid exercises very little influence on the result." § 837.

Apparent oceanic tides.—If the earth were a fluid of density 5·5 and the ocean rested upon the heavier liquid, both would yield to the attraction of the moon almost equally and there would be no considerable apparent tide. If the earth were a perfectly rigid mass and always turned the same exposure toward the moon, the water directly under the moon would rise $\frac{5}{2} \cdot 1\frac{3}{4}$ feet or $4\frac{3}{8}$ feet higher than it would stand at an angular distance of 90° from the moon. The solar tides are about 1/2 of the lunar tides. The actual total amplitude of the tides at oceanic ports is usually from 2 to 3 feet. Were the actual tides equal to the equilibrium tides, it would be perfectly easy to state the approximate rigidity of the earth. The real tides are greatly affected by the rotation of the earth, the obstruction of the continents, and the inequalities of the bottom. They are not static but dynamic. Nevertheless the fact that these are apparent tides shows that the earth possesses some rigidity, and the fact their amplitude is so considerable compared with the equilibrium amplitude shows that this rigidity is great. Thus if the globe were as rigid as glass, the apparent tides would be only 2/5 of the real tides or in other words perfect rigidity would then increase the apparent tides to 5/2 of their present amplitude, so that the amplitude of the tides at oceanic stations would increase to from 5 to $8\frac{1}{2}$ feet and would be much greater than the lunar equilibrium tides. If the rigidity is really that of brass, infinite rigidity would raise the oceanic tides to from 4 to 6 feet. If the earth were as strong as steel these tides would be raised by perfect rigidity to from $2\frac{1}{2}$ to $4\frac{1}{4}$ feet.

Messrs. Thomson and Darwin have examined the tidal theories and observations with especial reference to the question of rigidity. In 1888, the latter wrote, "it cannot be admitted that perfect rigidity of the earth would augment the tides in the proportion of 5 to 2, although they might perhaps be augmented in the proportion of 4 to 3."* The context shows that Darwin includes tides of short period as well as those of long period in this statement.

Rigidity of the earth.—The conditions affecting the tides of short period are so complex that no more information can be derived from them than is contained in the above quotation, which amounts to the statement that the earth's modulus of rigidity must exceed 24,400 kilograms per square centimeter by some indefinite amount. But the tides of long period must conform much more nearly to the equilibrium values than do those of short period. Were the period so long that the kinetic energy of the ebb and flow currents would be sensibly dissipated, or were the waters practically motionless at high

* Encyc. Brit., Article Tides.

water and low water, the long period tides would coincide with those of equilibrium and then observation would give a reasonably accurate value of the rigidity of the earth. Laplace supposed that the fortnightly tides did sensibly coincide with the equilibrium heights. On this hypothesis Darwin discussed the observations and found that the amplitude of the lunar fortnightly and monthly tides is about 2/3 of the equilibrium height. This would correspond to the rigidity of steel. Later however he came to the conclusion that these periods are too short to bring the waters to rest and that the result is thus vitiated.

It is certain, however, as Mr. Darwin wrote me in November, 1889, "that these tides will not be large on a rigid earth and hence the investigation remains as a general confirmation of the rigidity of the earth, without giving any quantitative evaluation of its amount."*

If the earth had the rigidity of brass the semidiurnal rise and fall of the land would be half the amplitude of the real tides and would equal the apparent tides. Were the continents uniform in lithological composition and elevation the tidal wave might pass under our feet unnoticed, as it does under a ship at sea. But it seems to me that at points near abrupt changes of density, as along the foot of a great mountain range or near the edge of a great oceanic abyss such as lies close to the Pacific Coast, a semidiurnal rise and fall of two or three feet would surely manifest itself in differential displacements.

Solidity of the earth.—The earth is thus a very rigid body. It does not follow as a matter of logical necessity that it is solid, because under certain circumstances a fluid in motion may act like a rigid mass toward certain external forces. On this subject it can only be said that, though the matter has been considered, physicists have thus far been unable to devise any system which will account for the effective rigidity of the earth as displayed in the tidal phenomena in this manner. No such explanation would be satisfactory unless the fluid were treated as viscous.

* Mr. Osmond Fisher quotes Mr. Love in the Proc. London Math. Soc., vol. xix, 1888, p. 206, as reaching the result that only the fortnightly tides can settle the question. If there is such a tide "we shall be entitled to say that the tidal effective rigidity of the earth is too great to allow us to suppose it to consist of a liquid mass covered with a thin solid crust." But, he says, "the Tidal Committee of the British Association appears to be still doubtful whether there really is an appreciable fortnightly tide." Mr. Darwin is or has been a member of the Brit. Assoc. committees on tides (excepting one to promote tidal observations in Canada) and has usually written the reports of these committees. I can find no evidence in these reports of any such doubt as is mentioned. Darwin speaks of these tides as "distinctly sensible" in Proc. R. S., Nov. 25, 1886, and as being 2/3 of the equilibrium height in Encyc. Brit., Art. Tides, 1888. That he is still confident of their existence is clear from the extract from his letter given in the text.

The average rigidity of the rocks exposed at the surface of the earth is certainly much less than that which a continuous mass of glass or brass would exhibit and there is thus a distinct difficulty in accounting for so high a rigidity as the earth evinces, even on the theory that the globe is solid. On this point, however, certain experimental results of Dr. William Hallock are very instructive. He subjected wax and paraffin to a pressure of 96,000 pounds per square inch in a horizontal steel cylinder; and on the top of these substances he placed small silver coins. The coins were forced against the tube with such violence as to leave in the steel impressions which could be felt as well as seen.* Thus substances which at one atmosphere show trifling rigidity, may develop a rigidity comparable with that of steel at pressures such as are to be found at about 15 miles below the surface of the earth.

On the other hand it is at first surprising to learn that a globe of the size of the earth and as rigid as the best drawn brass, would yield to the attraction of a distant body half as freely as if it were fluid. But it is a matter of common observation that bodies of large dimensions are not so strong as small ones in proportion to their size. Thus a bar of steel of the size of a needle held horizontally by one end does not bend even sensibly and the strain is far within the elastic limit; while a bar of the same relative dimensions, but 1000 feet long, evidently could not be supported in a similar manner. In fact, when two bodies are geometrically similar, the strength varies as the squares of the corresponding dimensions, while the mass varies as the cubes of these dimensions. When we come to deal with a mass like the earth, containing about 11×10^{20} cubic metres each weighing 5,500 kilograms, it therefore is not strange after all that it presents but a feeble resistance to external forces.

The opposition which some geologists have made to the theory of the solidity of the earth is of course based upon the hypothesis that geological phenomena cannot be accounted for on the theory of solidity. So far as the contortion of strata is concerned, there is no conflict between geology and physics. Time enters into the expression of viscosity; and the fact that the earth behaves as a rigid mass to a force which changes its direction by 360° in 24 hours is not inconsistent with great plasticity under the action of small forces which maintain their direction for ages. For a considerable number of years I have constantly had the theory of the earth's solidity in mind while making field observations on upheaval and subsidence, with the result that to my thinking, the phenomena

* This Journal, vol. xxxiv, 1887, p. 277.

are capable of much more satisfactory explanation on a solid globe than on an encrusted fluid one

The only considerable novelty in the foregoing paper is the proof that a simple strain spheroid affords an approximation to the deformation of an elastic globe sufficiently close to serve as a basis for Sir William Thomson's demonstration of the rigidity of the earth; but an endeavor has also been made to present the whole subject in a clear and elementary manner.

U. S. Geol. Survey, Washington, Feb. 1890.