

ART. X. — *Notes on the Electromagnetic Theory of Light*; by
J. WILLARD GIBBS. NO. III. *On the General Equations of
Monochromatic Light in Media of every degree of Transparency.*

1. THE last April and June numbers of this Journal* contain an investigation of the velocity of plain waves of light, in which they are regarded as consisting of solenoidal electrical fluxes in an indefinitely extended medium of uniform and very fine-grained structure. It was also supposed that the medium was perfectly transparent, although without discussion of the physical properties on which transparency depends, and that the electrical motions were not complicated by any distinctively magnetic phenomena.

In the present paper† the subject will be treated with more generality, so as to obtain the general equations of monochromatic light for media of every degree of transparency, whether sensibly homogeneous or otherwise, which have a very fine-grained molecular structure as measured by a wave-length of light. There will be no restriction with respect to magnetic influence, except that an oscillating magnetization of the medium will be excluded.‡

In order to conform as much as possible to the ordinary view of electrical phenomena,§ we shall not introduce at first

* See volume xxiii of this Journal, pages 262–275, and 460–476.

† This paper contains, with some additional developments, the substance of a communication to the National Academy of Sciences in November, 1882.

‡ Where a body capable of magnetization is subjected to the influence of light, (as when light is reflected from the surface of iron,) there are two simple hypotheses which present themselves with respect to the magnetic state of the body. One is that the magnetic forces due to the light are not of sufficient duration to allow the molecular changes which constitute magnetization to take place to any sensible extent. The other is that the magnetization has a constant ratio to the magnetic force without regard to its duration. We might easily make a more general hypothesis which would embrace both of those mentioned as extreme cases, and which would be irreproachable from a theoretical stand-point; but it would complicate our equations to a degree which would not be compensated by their greater generality, since no phenomena depending on such magnetization have been observed, so far as the writer is aware, or are likely to be, except in a very limited class of cases.

For the purposes of this paper, therefore, it has seemed better to exclude media capable of magnetization, except so far as the first mentioned hypothesis may be applicable. But it does not appear that this requires us to exclude cases in which the medium is subject to the influence of a permanent magnetic force, such as produces the phenomenon of the magnetic rotation of the plane of polarization.

§ It has, perhaps, retarded the acceptance of the electromagnetic theory of light that it was presented in connection with a theory of electrical action, which is probably more difficult to prove or disprove, and certainly presents more difficulties of comprehension, than the connection of optical and electrical phenomena, and which, as resting largely on *a priori* considerations, must naturally appear very differently to different minds. Moreover, the mathematical method by which the subject was treated, while it will remain a striking monument of its author's originality of thought, and profoundly modify the development of mathe-

the hypothesis of Maxwell that electrical fluxes are solenoidal.* Our results, however, will be such as to require us to admit the substantial truth of this hypothesis, if we regard the processes involved in the transmission of light as electrical.

With regard to the undetermined questions of electrodynamic induction, we shall adopt provisionally that hypothesis which appears the most simple, yet proceed in such a manner that it will be evident exactly how our results must be altered, if we prefer any other hypothesis.

Electrical quantities will be treated as measured in electromagnetic units.

2. We must distinguish, as before, between the *actual* electrical displacements, which are too complicated to follow in detail with analysis, and which in their minutiae elude experimental demonstration, and the displacements as *averaged* for spaces which are large enough to smooth out their minor irregularities, but not so large as to obliterate to any sensible extent those more regular features of the electrical motion, which form the subject of optical experiment. These spaces must therefore be large as measured by the least distances between molecules, but small as measured by a wave-length of light. We shall also have occasion to consider similar averages for other quantities, as electromotive force, the electrostatic potential, etc. It will be convenient to suppose that the space for which the average is taken is the same in all parts of the field,† say a sphere of uniform radius having its center at the point considered.

Whatever may be the quantities considered, such averages will be represented by the notation

$$[\quad]_{\text{Ave}}$$

If, then, ξ , η , ζ denote the components of the actual displacement at the point considered,

$$[\xi]_{\text{Ave}},$$

$$[\eta]_{\text{Ave}},$$

$$[\zeta]_{\text{Ave}}$$

will represent the average values of these components in the small sphere about that point. These average values we shall

mathematical physics, must nevertheless, by its wide departure from ordinary methods, have tended to repel such as might not make it a matter of serious study.

*A flux is said to be *solenoidal* when it satisfies the conditions which characterize the motion of an incompressible fluid,—in other words, if u , v , w are the rectangular components of the flux, when

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

and the normal component of the flux is the same on both sides of any surfaces of discontinuity which may exist.

† This is rather to fix our ideas, than on account of any mathematical necessity. For the space for which the average is taken may in general be considerably varied without sensibly affecting the value of the average.

treat as functions of the coördinates of the center of the sphere and of the time, and may call them, for brevity, the *average values* of ξ, η, ζ . But however they may be designated, it is essential to remember that it is a space-average for a certain very small space, and never a time-average, that is intended.

The object of this paper will be accomplished when we have expressed (explicitly or implicitly) the relations which subsist between the values of $[\xi]_{\text{Ave}}, [\eta]_{\text{Ave}}, [\zeta]_{\text{Ave}}$, at different times and in different parts of the field,—in other words, when we have found the conditions which these quantities must satisfy as functions of the time and the coördinates.

3. Let us suppose that luminous vibrations of any one period* are somewhere excited, and that the disturbance is propagated through the medium. The motions which are excited in any part of the medium, and the forces by which they are kept up, will be expressed by harmonic functions of the time, having the same period,† as may be proved by the single principle of the superposition of motions, quite independently of any theory of the constitution of the medium, or of the nature of the motions, as electrical or otherwise. This is equally true of the actual motions, and of the averages which we are to consider. We may therefore set

$$\left. \begin{aligned} [\xi]_{\text{Ave}} &= a_1 \cos \frac{2\pi}{p}t + a_2 \sin \frac{2\pi}{p}t, \\ \text{etc.,} \end{aligned} \right\} \quad (1)$$

where t denotes the time, p the period, and a_1, a_2 , functions of the coördinates. It follows that

$$\left. \begin{aligned} [\ddot{\xi}]_{\text{Ave}} &= -\frac{4\pi^2}{p^2} [\xi]_{\text{Ave}} \\ \text{etc.} \end{aligned} \right\} \quad (2)$$

* There is no real loss of generality in making the light monochromatic, since in every case it may be divided into parts, which are separately propagated, and each of which is monochromatic to any required degree of approximation.

† It is of course possible that the expressions for the forces and displacements should have constant terms. But these will disappear, if the displacements are measured from the state of equilibrium about which the system vibrates, and we leave out of account in measuring the forces (and the electrostatic potential) that which would belong to the system in the state of equilibrium. To prevent misapprehension, it should be added that the term *electrical displacement* is not used in the restricted sense of *dielectric displacement* or *polarization*. The variation of the electrical displacement, as the term is used in this paper, constitutes what Maxwell calls the total motion of electricity or true current, and what he divides into two parts, which he distinguishes as the current of conduction and the variation of the electrical displacement. Such a division of the total motion of electricity is not necessary for the purposes of this paper, and the term displacement is used with reference to the total motion of electricity in a manner entirely analogous to that in which the term is ordinarily used in the theory of wave-motion.

4. Now, on the electrical theory, these motions are excited by electrical forces, which are of two kinds, distinguished as electrostatic and electrodynamic. The electrostatic force is determined by the electrostatic potential. If we write q for the actual value of the potential, and $[q]_{\text{Ave}}$ for its value as averaged in the manner specified above, the components of the actual electrostatic force will be

$$-\frac{dq}{dx}, \quad -\frac{dq}{dy}, \quad -\frac{dq}{dz};$$

and for the average values of these components in the small spaces described above we may write

$$-\frac{d[q]_{\text{Ave}}}{dx}, \quad -\frac{d[q]_{\text{Ave}}}{dy}, \quad -\frac{d[q]_{\text{Ave}}}{dz},$$

for it will make no difference whether we take the average before or after differentiation.

5. The electrodynamic force is determined by the acceleration of electrical flux in all parts of the field, but physicists are not entirely agreed in regard to the laws by which it is determined. This difference of opinion is however of less importance, since it will not affect the result if electrical fluxes are always solenoidal. According to the most simple law, the components of the force are given by the volume-integrals

$$-\iiint \frac{\ddot{\xi}}{r} dv, \quad -\iiint \frac{\ddot{\eta}}{r} dv, \quad -\iiint \frac{\ddot{\zeta}}{r} dv,$$

where dv represents an element of volume, and r the distance of this element from the point for which the value of the electromotive force is to be determined. In other words, the components of the force at any point are determined from the components of acceleration in all parts of the field by the same process by which (in the theories of gravitation, etc.) the value of the potential at any point is determined from the density of matter in all parts of space, except that the sign is to be reversed. Adopting this law provisionally, at least, we may express it by saying that the components of electrodynamic force are equal to the potentials taken negatively of the components of acceleration of electrical flux. And we may write, for brevity,

$$-\text{Pot } \ddot{\xi}, \quad -\text{Pot } \ddot{\eta}, \quad -\text{Pot } \ddot{\zeta},$$

for the components of force, using the symbol *Pot* to denote the operation by which the potential of a mass is derived from its density. For the average values of these components in the small spaces defined above, we may write

$$-\text{Pot } [\ddot{\xi}]_{\text{Ave}}, \quad -\text{Pot } [\ddot{\eta}]_{\text{Ave}}, \quad -\text{Pot } [\ddot{\zeta}]_{\text{Ave}}$$

since it will make no difference whether we take the average before or after the operation of taking the potential.

6. If we write X, Y, Z for the components of the total electromotive force (electrostatic and electrodynamic), we have

$$\left. \begin{aligned} [X]_{\text{Ave}} &= -\text{Pot} [\ddot{\xi}]_{\text{Ave}} - \frac{d[q]_{\text{Ave}}}{dx}, \\ \text{etc.;} \end{aligned} \right\} \quad (3)$$

or by (2)

$$\left. \begin{aligned} [X]_{\text{Ave}} &= \frac{4\pi^2}{p^2} \text{Pot} [\xi]_{\text{Ave}} - \frac{d[q]_{\text{Ave}}}{dx}, \\ \text{etc.} \end{aligned} \right\} \quad (4)$$

It will be convenient to represent these relations by a vector notation. If we represent the displacement by $\mathbf{U}$, and the electromotive force by $\mathbf{E}$, the three equations of (3) will be represented by the single vector equation

$$[\mathbf{E}]_{\text{Ave}} = -\text{Pot} [\ddot{\mathbf{U}}]_{\text{Ave}} - \nabla[q]_{\text{Ave}}, \quad (5)$$

and the three equations of (4) by the single vector equation

$$[\mathbf{E}]_{\text{Ave}} = \frac{4\pi^2}{p^2} \text{Pot} [\mathbf{U}]_{\text{Ave}} - \nabla[q]_{\text{Ave}}, \quad (6)$$

where, in accordance with quaternionic usage, $\nabla[q]_{\text{Ave}}$ represents the vector which has for components the derivatives of $[q]_{\text{Ave}}$ with respect to rectangular coördinates. The symbol *Pot* in such a vector equation signifies that the operation which is denoted by this symbol in a scalar equation is to be performed upon each of the components of the vector.

7. We may here observe that if we are not satisfied with the law adopted for the determination of electrodynamic force, we have only to substitute for $-\text{Pot}$ in these vector equations, and in those which follow, the symbol for the operation, whatever it may be, by which we calculate the electrodynamic force from the acceleration.* For the operation must be of such a character, that if the acceleration consist of any number of parts, the force due to the whole acceleration will be the resultant of the forces due to the separate parts. It will evidently make no difference whether we take an average before or after such an operation.

8. Let us now examine the relation which subsists between the values of $[\mathbf{E}]_{\text{Ave}}$ and $[\mathbf{U}]_{\text{Ave}}$ for the same point, that is, between the average electromotive force and the average displacement in a small sphere with its center at the point consid-

*The same would not be true of the corresponding scalar equations, (3) and (4). For one component of the force might depend upon all the components of acceleration. Such is in fact the case with the law of electromotive force proposed by Weber.

ered. We have already seen that the forces and the displacements are harmonic functions of the time having a common period.

A little consideration will show that if the average electromotive force in the sphere is given as a function of the time, the displacements in the sphere, both average and actual, must be entirely determined. Especially will this be evident, if we consider that since we have made the radius of the sphere very small in comparison with a wave-length, the average force must have sensibly the same value throughout the sphere, (that is, if we vary the position of the center of the sphere for which the average is taken by a distance not greater than the radius, the value of the average will not be sensibly affected,) and that the difference of the actual and average force at any point is entirely determined by the motions in the immediate vicinity of that point. If, then, certain oscillatory motions may be kept up in the sphere under the influence of electrostatic and electrodynamic forces due to the motion in the whole field, and if we suppose the motions in and very near that sphere to be unchanged, but the motions in the remoter parts of the field to be altered, only not so as to affect the average resultant of electromotive force in the sphere, the actual resultant of electromotive force will also be unchanged throughout the sphere, and therefore the motions in the sphere will still be such as correspond to the forces.

Now the average displacement is a harmonic function of the time having a period which we suppose given. It is therefore entirely determined for the whole time the vibrations continue by the values of the six quantities

$$[\xi]_{\text{Ave}}, [\eta]_{\text{Ave}}, [\zeta]_{\text{Ave}}, [\dot{\xi}]_{\text{Ave}}, [\dot{\eta}]_{\text{Ave}}, [\dot{\zeta}]_{\text{Ave}}$$

at any one instant. For the same reason the average electromotive force is entirely determined for the whole time by the values of the six quantities

$$[X]_{\text{Ave}}, [Y]_{\text{Ave}}, [Z]_{\text{Ave}}, [\dot{X}]_{\text{Ave}}, [\dot{Y}]_{\text{Ave}}, [\dot{Z}]_{\text{Ave}}$$

for the same instant. The first six quantities will therefore be functions of the second, and the principle of the superposition of motions requires that they shall be homogeneous functions of the first degree. And the second six quantities will be homogeneous functions of the first degree of the first six. The coefficients by which these functions are expressed will depend upon the nature of the medium in the vicinity of the point considered. They will also depend upon the period of vibration, that is, upon the color of the light.*

*The relations between the displacements in one of the small spaces considered and the average electromotive force is mathematically analogous to the relation between the displacements in a system of a high degree of complexity and certain forces exerted from without, which are harmonic functions of the time and under

We may therefore write in vector notation

$$[\mathbf{E}]_{\text{Ave}} = \Phi[\mathbf{U}]_{\text{Ave}} + \Psi[\dot{\mathbf{U}}]_{\text{Ave}} \quad (7)$$

where Φ and Ψ denote linear functions.*

The optical properties of media are determined by the form of these functions. But all forms of linear functions would not be consistent with the principle of the conservation of energy.

In media which are more or less opaque, and which therefore absorb energy, Ψ must be of such a form that the function always makes an acute angle (or none) with the independent variable. In perfectly transparent media Ψ must vanish, unless the function is at right angles to the independent variable. So far as is known, the last occurs only when the medium is subject to magnetic influence. In perfectly transparent media, the principle of the conservation of energy requires that Φ should be self-conjugate, i. e., that for three directions at right angles to one another, the function and independent variable should coincide in direction.

In all isotropic media not subject to magnetic influence, it is probable that Φ and Ψ reduce to numerical coefficients, as is certainly the case with Φ for transparent isotropic media.

9. Comparing the two values of $[\mathbf{E}]_{\text{Ave}}$, we have

$$\frac{4\pi^2}{p^2} \text{Pot} [\mathbf{U}]_{\text{Ave}} - \nabla[q]_{\text{Ave}} = \Phi[\mathbf{U}]_{\text{Ave}} + \Psi[\dot{\mathbf{U}}]_{\text{Ave}} \quad (8)$$

This equation, in connection with that by which we express the solenoidal character of the displacements, if we regard them as necessarily solenoidal, or in connection with that which expresses the relation between the electrostatic potential and the displacements, if we reject the solenoidal hypothesis, may be regarded as the general equation of the vibrations of monochromatic light, considered as oscillating electrical fluxes. For the symbol *Pot*, however, we must substitute the symbol representing the operation by which electromotive force is calculated from acceleration of flux, with the negative sign, if we are not satisfied with the law provisionally adopted.

It is important to observe that the existence of molecular vibrations of ponderable matter, due to the passage of light through the medium, will not affect the reasoning by which this equation has been established, provided that the nature

the influence of which the system vibrates. The ratio of the displacements to the forces will in general vary with the period, and may vary very rapidly.

An example in which these functions vary very rapidly with the period is afforded by the phenomena of selective absorption and abnormal dispersion.

* A vector is said to be a linear function of another, when the three component of the first are homogeneous functions of the first degree of the three component of the second.

and intensity of these vibrations in any small part of the medium (as measured by a wave-length) are entirely determined by the electrical forces and motions in that part of the medium. But the equation would not hold in case of molecular vibrations due to magnetic force. Such vibrations would constitute an oscillating magnetization of the medium, which has already been excluded from the discussion.

The supposition which has sometimes been made,* that electricity possesses a certain mass or inertia, would not at all affect the validity of the equation.

10. The equation may be reduced to a form in some respects more simple by the use of the so-called imaginary quantities. We shall write ι for $\sqrt{-1}$. If we differentiate with respect to the time, and substitute $-\frac{4\pi^2}{p^2}[\mathbf{U}]_{\text{Ave}}$ for $[\ddot{\mathbf{U}}]_{\text{Ave}}$, we obtain

$$\frac{4\pi^2}{p^2} \text{Pot } [\dot{\mathbf{U}}]_{\text{Ave}} - \nabla[\dot{\mathbf{q}}]_{\text{Ave}} = \Phi[\dot{\mathbf{U}}]_{\text{Ave}} - \frac{4\pi^2}{p^2} \Psi[\mathbf{U}]_{\text{Ave}}$$

If we multiply this equation by ι , either alone or in connection with any real factor, and add it to the preceding, we shall obtain an equation which will be equivalent to the two of which it is

formed. Multiplying by $\iota - \frac{p\iota}{2\pi}$ and adding, we have

$$\begin{aligned} \frac{4\pi^2}{p^2} \text{Pot} \left([\mathbf{U}]_{\text{Ave}} - \iota \frac{p}{2\pi} [\dot{\mathbf{U}}]_{\text{Ave}} \right) - \nabla \left([\mathbf{q}]_{\text{Ave}} - \iota \frac{p}{2\pi} [\dot{\mathbf{q}}]_{\text{Ave}} \right) \\ = \left(\Phi + \iota \frac{2\pi}{p} \Psi \right) \left([\mathbf{U}]_{\text{Ave}} - \iota \frac{p}{2\pi} [\dot{\mathbf{U}}]_{\text{Ave}} \right). \end{aligned}$$

If we set

$$\mathbf{W} = [\mathbf{U}]_{\text{Ave}} - \iota \frac{p}{2\pi} [\dot{\mathbf{U}}]_{\text{Ave}}, \quad (9)$$

$$\mathbf{Q} = [\mathbf{q}]_{\text{Ave}} - \iota \frac{p}{2\pi} [\dot{\mathbf{q}}]_{\text{Ave}}, \quad (10)$$

$$\Theta = \Phi + \iota \frac{2\pi}{p} \Psi, \quad (11)$$

our equation reduces to

$$\frac{4\pi^2}{p^2} \text{Pot } \mathbf{W} - \nabla \mathbf{Q} = \Theta \mathbf{W}. \quad (12)$$

In this equation Θ denotes a complex linear vector function, *i. e.*, a vector function of which the X-, Y-, and Z-components are expressed in terms of the X-, Y-, and Z-components of the independent variable by means of coefficients of the form $a + \iota b$. $\mathbf{W}$ is a bi-vector of which the real part represents the

* See Weber, Abhandl. d. K. Sächs. Gesellsch. d. Wiss., vol. vi, p. 593-597; Lorberg, Crelle's Journal, vol. lxi, p. 55.

averaged displacement $[U]_{\text{Ave}}$, and the coefficient of ι the rate of increase of the same multiplied by a constant factor. This bi-vector therefore represents the average state of a small part of the field both with respect to position and velocity. We may also say that the coefficient of ι in $\mathbf{W}$ represents the value of the averaged displacement $[U]_{\text{Ave}}$ at a time one-quarter of a vibration earlier than the time principally considered.

11. It may serve to fix our ideas to see how $\mathbf{W}$ is expressed as a function of the time. We may evidently set

$$[U]_{\text{Ave}} = \mathbf{A}_1 \cos \frac{2\pi}{p} t + \mathbf{A}_2 \sin \frac{2\pi}{p} t$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are vectors representing the amplitudes of the two parts into which the vibration is resolved. Then

$$\frac{p}{2\pi} [\dot{U}]_{\text{Ave}} = -\mathbf{A}_1 \sin \frac{2\pi}{p} t + \mathbf{A}_2 \cos \frac{2\pi}{p} t,$$

and

$$[U]_{\text{Ave}} - \iota \frac{p}{2\pi} [\dot{U}]_{\text{Ave}} = (\mathbf{A}_1 - \iota \mathbf{A}_2) \left(\cos \frac{2\pi}{p} t + \iota \sin \frac{2\pi}{p} t \right);$$

that is, if we set $\mathbf{A} = \mathbf{A}_1 - \iota \mathbf{A}_2$,

$$\mathbf{W} = \mathbf{A} e^{\frac{2\pi \iota t}{p}}. \quad (13)$$

In like manner we may obtain

$$\mathbf{Q} = g e^{\frac{2\pi \iota t}{p}}, \quad (14)$$

where g is a bi-scalar, or complex quantity of ordinary algebra. Substituting these values in (12), and cancelling the common factor containing the time, we have

$$\frac{4\pi^2}{p^2} \text{Pot } \mathbf{A} - \nabla g = \Theta \mathbf{A}. \quad (15)$$

Our equation is thus reduced to one between $\mathbf{A}$ and g , and may easily be reduced to one in $\mathbf{A}$ alone.* Now $\mathbf{A}$ represents six numerical quantities, (viz: the three components of $\mathbf{A}_1$, and the three of $\mathbf{A}_2$), which may be called the six components of amplitude. The equation, therefore, substantially represents the relations between the six components of amplitude in different parts of the field.† The equation is, however, not really

* The terms $\nabla \mathbf{Q}$, ∇g are allowed to remain in these equations, because the best manner of eliminating them will depend somewhat upon our admission or rejection of the solenoidal hypothesis.

† The representation of the six components of amplitude by a single letter should not be regarded as an analytical artifice. It only leaves undivided in our notation that which is undivided in the nature of things. The separation of the six components of amplitude is artificial, in that it introduces arbitrary elements into the discussion, viz: the directions of the axes of the coördinates, and the zero of time.

different from (12), since $\mathbf{A}$ and g are only particular values of $\mathbf{W}$ and $\mathbf{Q}$.

12. From the general equation given above (8, 12, or 15), in connection with the solenoidal hypothesis, we may easily derive the laws of the propagation of plane waves in the interior of a sensibly homogeneous medium, and the laws of reflection and refraction at surfaces between such media. This has been done by Maxwell,* Lorentz,† and others,‡ with fundamental equations more or less similar.

The method, however, by which the fundamental equation has been established in this paper seems free from certain objections which have been brought against the ordinary form of the theory. As ordinarily treated, the phenomena are made to depend entirely on the inductive capacity and the conductivity of the medium, in a manner which may be expressed by the equation

$$[\mathbf{U}]_{\text{Ave}} = \left(\frac{\mathbf{K}}{4\pi} - \frac{p^2 \mathbf{C}}{4\pi^2} \frac{d}{dt} \right) \left(\frac{4\pi^2}{p^2} \text{Pot } [\mathbf{U}]_{\text{Ave}} - \nabla [\mathbf{q}]_{\text{Ave}} \right), \quad (16)$$

which will be equivalent to (12), if

$$\mathbf{W} = \left(\frac{\mathbf{K}}{4\pi} - i \frac{p \mathbf{C}}{2\pi} \right) \left(\frac{4\pi^2}{p^2} \text{Pot } \mathbf{W} - \nabla \mathbf{Q} \right), \quad (17)$$

where $\mathbf{K}$ and $\mathbf{C}$ denote in the most general case the linear vector functions, but in isotropic bodies the numerical coefficients, which represent inductive capacity and conductivity. By a simple transformation [see (9) and (10)], this equation becomes

$$\Theta^{-1} = \frac{\mathbf{K}}{4\pi} - i \frac{p \mathbf{C}}{2\pi}, \quad (18)$$

where Θ^{-1} represents the function inverse to Θ .

Now, while experiment appears to verify the existence of such a law as is expressed by equation (12), it does not show that Θ has the precise form indicated by equation (16). In other words, experiment does not satisfactorily verify the relations expressed by (16) and (17), if $\mathbf{K}$ and $\mathbf{C}$ are understood to be the operators (or, in isotropic bodies, the numbers) which represent induction capacity and conductivity in the ordinary sense of the terms.

* Phil. Trans., vol. clv (1865), p. 459, or *Treatise on Electricity and Magnetism*, Chap. XX.

† Schlämilch's Zeitschrift, vol. xxii, pp. 1-30 and 205-219; xxiii, pp. 197-210.

‡ See Fitzgerald, Phil. Trans., vol. clxxi, p. 691; J. J. Thomson, Phil. Mag., V, vol. ix, p. 284; Rayleigh, Phil. Mag., V, vol. xii, p. 81.

That the electromagnetic theory of light gives the conditions relative to the boundary of different media, which are required by the phenomena of reflection and refraction, was first shown by Helmholtz. See Crelle's Journal, vol. lxxii (1870), p. 57.

The discrepancy is most easily shown in the most simple case, when the medium is isotropic and perfectly transparent, and Θ reduces to a numerical quantity. The square of the velocity of plane waves is then equal to $\frac{\Theta}{4\pi}$, and equation (18)

would make it independent of the period; that is, would give no dispersion of colors. The case is essentially the same in transparent bodies which are not isotropic.*

The case is worse with metals, which are characterized electrically by great conductivity, and optically by great opacity. In their papers cited above, Lorentz and Rayleigh have observed that the experiments of Jamin on the reflection of light from metallic surfaces would often require, as ordinarily interpreted on the electro-magnetic theory, a negative value for the inductive capacity of the metal. This would imply that the electrical equilibrium in the metal is unstable. The objection, therefore, is essentially the same as that which Lord Rayleigh had previously made to Cauchy's theory of metallic reflection, viz: that the apparent mechanical explanation of the phenomena is illusory, since the numerical values given by experiment as interpreted on Cauchy's theory would involve an unstable equilibrium of the ether in the metal.†

13. All this points to the same conclusion—that the ordinary view of the phenomena is inadequate. The object of this paper will be accomplished, if it has been made clear, how a point of view more in accordance with what we know of the molecular constitution of bodies will give that part of the ordinary theory which is verified by experiment, without including that part which is in opposition to observed facts.‡

* See note to the first paper of Lorentz, cited above, *Schlömilch*, vol. xxii, p. 23.

† See *Phil. Mag.*, IV, vol. xliii (1872), p. 321.

‡ The consideration of the processes which we may suppose to take place in the smallest parts of a body through which light is transmitted, farther than is necessary to establish the general equation given above, is foreign to the design of this paper. Yet a word may be added with respect to the difficulties signalized in the ordinary form of the theory. The comparatively simple case of a perfectly transparent body has been examined more in detail in one of the papers already cited, where there is given an explanation of the dispersion of colors from the point of view of this paper. It is there shown that the effect of the non-homogeneity of the body in its smallest parts is to add a term to the expression for the kinetic energy of electrical waves, which for an isotropic body may be roughly described as similar to that which would be required if the electricity had a certain mass or inertia. (See especially §§ 7, 9 and 12, pages 266 ff. of volume xxiii of this Journal.) The same must be true of media of any degree of opacity. Now the difficulty with the optical properties of the metals is that the real part of Θ (or Θ^{-1}) is in some cases negative. This implies that at a moment of greatest displacement the electromotive force is in the direction opposite to the displacement, instead of having the same direction, as in transparent isotropic bodies. Now a certain part of the electromotive force must be required to oppose the apparent inertia, and another part to oppose the electrical elasticity of the medium. These parts of the force must have opposite directions. In transparent

While the writer has aimed at a greater degree of rigor than is usual in the establishment of the fundamental equation of monochromatic light, it is not claimed that this equation is absolutely exact. The contrary is evident from the fact that the equation does not embrace the phenomena which characterize such circularly polarizing bodies as quartz. This, however, only implies the neglect of extremely small quantities—very small, for example, as compared with those which determine the dispersion of colors. In one of the papers already cited,* the case of a perfectly transparent body is treated with a higher degree of approximation, so as to embrace the phenomena in question.