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ART. I.—*On the Inequalities in the Moon's Motion produced by the Oblateness of the Earth;* by Professor J. N. STOCKWELL.

HAVING given in the November number of this Journal, an account of a secular inequality in the moon's motion, arising from the oblateness of the earth, I propose to give, in the present number, a somewhat detailed account of the principal periodic inequalities in the motions of the moon arising from the same cause. And I would here state that a careful study of the effect of the earth's ellipticity on the motions of the moon, through the medium of analysis, has presented some of the most curious and interesting cases of perturbation to be found in physical astronomy. The principal inequalities to which I would call attention, are chiefly remarkable as being the product of a number of important, though mutually antagonistic forces, the resultant of which, on any given coördinate of the moon, is of far less importance than would arise from the undisturbed operation of any one of the constituent forces; and I do not remember any cases of perturbation in the moon's motion, arising from the sun's attraction, in which the effect of the force on any given coördinate is so completely neutralized by the action of the same force on some of the other elements of motion. The analysis which I have employed has presented the results in such shape as to show: *First*, the separate effect of the earth's oblateness on the motions of the moon; and *Second*, the combined effect resulting from the earth's figure and the sun's attraction. It is from these two points of view that I shall now consider the subject.

In the paper referred to above, it was shown that the motion of a body moving in a circular orbit in the plane of the equator would be uniform; and that the motion in a circular orbit which was inclined to the equator would be subject to inequalities, the magnitude of which would depend on the inclination of the orbit to the equator. Now the ecliptic is inclined to the equator at an angle of 23° 27′, and the moon's orbit is inclined to the ecliptic at an angle of 5° 8′. If, then, the ascending node of the moon's orbit is at the vernal equinox, the inclination of the moon's orbit to the equator will be 28° 35′; and she will attain to that degree of declination twice during each sidereal revolution. In this position of the node, the inequalities of the earth's attraction on the moon attain their maximum values. Suppose, now, that the ascending node is at the autumnal equinox. It is evident that the inclination of the orbit to the equator will be only 18° 19′, and that she will only reach that declination twice during each revolution. The inequalities of the earth's attraction on the moon, in this last position of the node, will evidently be at their minimum values. As the node passes from one equinox to the other, it is plain that the inequalities of the earth's attraction on the moon pass through all the changes of value to which they are at any time subject. Now the moon's node makes a complete revolution on the ecliptic in a period of 18·6 years; consequently all the inequalities of the moon's motion arising from the oblateness of the earth will effect a complete restoration in that period. Since the motion of the moon's node is caused by the sun's attraction, we must, in the calculation of the separate effect of the spheroidal form of the earth on the moon's motion, neglect the sun's attraction and regard the elements of the moon's orbit as constant, except so far as they are affected by the form of the earth itself. From these general considerations it follows that the moon's declination, on which the inequalities of the earth's attractive force depends, is affected by two conditions: namely, the longitude of the moon and the longitude of the node. I shall therefore in the present paper consider only the inequalities which depend on these two elements, either separately or in combination.

In the calculations which I have made, the oblateness of the earth has been taken as $\frac{1}{300}$; and in the few equations which are given, the symbols have the following significations: a and nt denote the moon's mean distance and mean longitude; v and θ denote the moon's true longitude and latitude. γ and ☊ denote the inclination and longitude of node of moon's orbit; ε denotes the obliquity of the ecliptic, and D denotes the *mean* radius of the earth; ρ and φ denote the oblateness of the earth, and the ratio of the centrifugal force to the gravity at the earth's equator; R denotes the disturbing function; and δ placed before a quantity denotes the variation of that quantity arising from the oblateness of the earth. I also put for brevity

$$\beta = \{\rho - \tfrac{1}{2}\varphi\}\frac{D^2}{a^2}\sin\varepsilon\cos\varepsilon. \tag{1}$$

I now give the equations which express the variations of the elements and coördinates of the moon, and which are independent of the sun's action. In the first place I find that the position of the node and inclination of the orbit are affected by the following inequalities:

$$\delta\Omega = \tfrac{1}{2}\frac{\beta}{\gamma}\sin(2nt - \Omega) = +0''{\cdot}185\sin(2nt - \Omega), \tag{2}$$

$$\delta\gamma = \tfrac{1}{2}\beta\cos(2nt - \Omega) = +0''{\cdot}0165\cos(2nt - \Omega). \tag{3}$$

These two inequalities give rise to the following inequality in the moon's latitude:

$$\delta\theta = -\tfrac{1}{2}\beta\sin nt = -0''{\cdot}0165\sin nt. \tag{4}$$

But I find that the direct action of the earth on the moon produces the following inequality:

$$\delta\theta = +\tfrac{1}{2}\beta\sin nt, \tag{5}$$

The sum of these two inequalities gives $\delta\theta = 0$; whence it follows that there is no equation of the above form in the moon's latitude arising from the oblateness of the earth. In other words, the figure of the earth does not cause the moon to depart from the plane of the great circle in which its orbit is situated.

The above values of $\delta\Omega$ and $\delta\gamma$ also give the following inequality in the moon's longitude:

$$\delta v = +\tfrac{1}{4}\beta\gamma\sin\Omega = +0''{\cdot}00074\sin\Omega). \tag{6}$$

The direct action of the earth on the moon produces the following inequalities in the longitude:

$$\left.\begin{aligned}\delta v &= \tfrac{1}{12}\beta\tan\varepsilon\sin 2nt - \tfrac{1}{12}\beta\gamma\sin(2nt - \Omega)\\ &= +0''{\cdot}0012\sin 2nt - 0''{\cdot}00025\sin(2nt - \Omega).\end{aligned}\right\} \tag{7}$$

These inequalities in longitude are so excessively small as to be entirely insensible. The sum of the coefficients in these three terms of δv amounts to only $0''{\cdot}0022$, a quantity not exceeding *fourteen feet* if measured on the moon's orbit. From this calculation it follows that, if the moon were entirely free from solar disturbance, the effect of the oblateness of the earth on its motions would be so small that it would never be detected by observation.

Let us now examine into the effect of a combination of solar disturbance with that arising from the earth's oblateness. Since we have supposed the moon's orbit to be circular, it is evident that the earth's attraction on the moon is at a maximum whenever the moon is in the equator, or twice during each revolution of the moon. We will now suppose that the longitude of the moon's node is 90°, and examine into the consequences that must take place while it retrogrades through a semi-circumference, or from $+90°$ to $-90°$.

When $\Omega = +90°$, the moon's orbit intersects the equator at a distance of 12° 45′ to the eastward of the equinox: and since the node retrogrades on the ecliptic about 1° 27′ during a sidereal revolution of the moon, it follows that the moon will arrive at the equator at a point a little to the westward of its previous crossing. In other words, the moon will make a complete revolution with respect to the center of force in a period somewhat shorter than the sidereal revolution. At the end of 9·8 years the longitude of the node will be $-90°$, and the orbit will intersect the equator at a distance of 12° 45′ to the westward of the equinox. Now the moon performs 124·3256 sidereal revolutions while the node is retrograding through an arc of 180°. But 124·3256 sidereal revolutions correspond to 124·3256 revolutions $+23° \; 21'$ with respect to the equator. Whence it appears that while the node is retrograding from $+90°$ to $-90°$, the time of revolution with respect to the equator is shorter on an average by $20^{m} \; 32^{s}$ than the sidereal revolution. It is plain that while the node is retrograding from $-90°$ through the autumnal equinox to $+90°$, the point of intersection of the orbit and equator will advance from $-12° \; 45'$ to $+12° \; 45'$, and the time of revolution of the moon with respect to the equator will exceed the time of the sidereal revolution by the same amount that it fell short of that quantity while retrograding through the other half of the orbit. It is also plain that the inclination of the orbit to the equator increases while the equatorial node is approaching the vernal equinox, at which point it is a maximum, and diminishing while it is receding from it. The constant retrograde motion of the ecliptic node of the moon's orbit, therefore, gives rise to a merely oscillatory motion of the equatorial node; and it is this pendulum-like motion of the equatorial node that gives rise to a number of inequalities in the moon's motion which I now proceed to consider; observing that the inequalities which are produced while the node is advancing are fully compensated by means of the retrograde motion which follows.

I now give the values of the perturbations of the elements and coördinates which I have obtained, as resulting from the motion of the moon's node, which is produced by the sun's attraction. I find

$$\left.\begin{aligned}\delta\Omega &= -\frac{\beta}{\alpha\gamma}\sin\Omega = +\,91''\cdot324\sin\Omega\\ \delta\gamma &= +\frac{\beta}{\alpha}\cos\Omega = -\,8''\cdot2233\cos\Omega\end{aligned}\right\}; \tag{8}$$

α being the ratio of the motion of the node to the moon's mean motion. These inequalities of the elements give rise to the following inequality in the latitude:

$$\delta\theta = \frac{\beta}{\alpha}\sin nt = -\,8''\cdot2233\sin nt. \tag{9}$$

This is exactly the same as LaPlace and subsequent investigators would have obtained had they used the same value of the earth's ellipticity. This inequality in the moon's latitude is equivalent to the supposition that the moon's orbit, instead of moving on the plane of the ecliptic with a constant inclination, moves, with the same condition, upon a plane passing always through the equinoxes, between the ecliptic and equator and inclined to the ecliptic by an angle which is equal to $\frac{\beta}{\alpha}$, as LaPlace has remarked.

I have not, however, been equally fortunate in reproducing the value of the inequality in the moon's longitude, which LaPlace and later investigators have obtained. I find as directly resulting from the motion of the moon's node, the following inequality in the longitude:

$$\delta v = -\frac{\beta}{\alpha^2}\gamma\sin\Omega = -\,184''\cdot3\sin\Omega. \tag{10}$$

But the preceding inequality in latitude gives rise to the two following inequalities in the longitude:

$$\left.\begin{aligned}\delta v &= +\frac{\beta}{\alpha_2}\gamma\sin\Omega - \tfrac{1}{2}\frac{\beta}{\alpha}\gamma\sin(2nt-\Omega)\\ &= +\,184''\cdot3\sin\Omega + 0''\cdot37\sin(2nt-\Omega)\end{aligned}\right\}. \tag{11}$$

The first of these two inequalities derived from the perturbations in latitude exactly cancels the preceding inequality in the longitude which is produced directly from the retrograde motion of the node; and we have as the resultant of the two forces,

$$\delta v = -\tfrac{1}{2}\frac{\beta}{\alpha}\gamma\sin(2nt-\Omega) = +\,0''\cdot37\sin(2nt-\Omega). \tag{12}$$

I find, however, that the radius vector of the moon's orbit is affected by the inequality

$$\delta v = 3\alpha\beta\gamma\cos\Omega, \tag{13}$$

and this produces the following inequality in the longitude,

$$\delta v = -6\frac{\beta}{\alpha}\gamma\sin\Omega = +\,4''\cdot443\sin\Omega, \tag{14}$$

while LaPlace found

$$\delta v = -\tfrac{19}{2}\frac{\beta}{\alpha}\,\gamma \sin \Omega = +\,7''{\cdot}03 \sin \Omega. \tag{15}$$

If, in the development of the inequalities depending on the oblateness of the earth we carry on the approximations so as to include terms of a higher order depending on the eccentricity and inclination of the orbit, we shall find two equations of sensible magnitude, having the same arguments as two empirical equations discovered by Airy about a third of a century ago. These equations depend on the arguments $nt - \omega - \Omega$, and $nt - \omega + \Omega$, in which ω denotes the longitude of the perigee. These arguments have periods of 27·4432 days, and 27·6661 days, respectively. The equation depending on the first of these arguments seems also to have been independently discovered quite recently, as an empirical equation, by Professor Newcomb, who attributes it to the attraction of some of the planets. From some calculations which I have made I am led to suspect that each of these equations has a value amounting to quite a large fraction of a second of arc; and I call attention to them here as being worthy of a more thorough investigation by astronomers.

It is but proper to add in this connection, that the mean motions of the perigee and node of the lunar orbit are affected by the oblateness of the earth; and are also affected by secular equations arising from the diminution of the obliquity of the ecliptic. The motions of the perigee and node which I have obtained agree in value with those obtained by LaPlace. I have therefore succeeded in reproducing exactly, by my method of computation, all the inequalities in the motions of the moon arising from the oblateness of the earth, which LaPlace discovered nearly a century ago, with the exception of the equation in longitude. The coefficient of LaPlace's equation exceeds the value which I have obtained, in the ratio of 19 to 12; and it has been a matter of surprise that two very dissimilar methods of computation should give so many results identically the same, and leave only a single one discordant. This has led me to make a critical examination of every step of LaPlace's calculation of this equation; and this examination has developed the fact that LaPlace has, in this instance, departed widely from the requirements of his own formulæ and methods; and that a correct calculation by his method gives a result identically the same as I have found by my own. I shall therefore now give the several steps of this examination, believing that it will not be without interest to the readers of this Journal. In this investigation it has been found convenient to use Bowditch's translation of the *Mécanique Céleste*, as the facilities for referring to any part of the work by means of the marginal numbers are much better than in the original. I shall also change the notation somewhat, putting ε for λ, and a for $g-1$ in some cases, and shall also put $f=1$. The numbers inclosed in brackets refer to the corresponding marginal numbers of the *Mécanique Céleste.*

The expression of the force R, [5362] becomes by using the value of β, which is given by equation (1) of this paper, and putting $f=1$,

$$R = 2\frac{a^2\beta}{r^3}s\sin v. \tag{16}$$

In this equation s denotes the tangent of the moon's latitude. This value of R gives the following values of the partial differential coefficients,

$$\left(\frac{dR}{dr}\right) = -6\frac{a^2\beta}{r^4}s\sin v, \tag{17}$$

$$\left(\frac{dR}{dv}\right) = 2\frac{a^2\beta}{r^3}s\cos v, \tag{18}$$

$$\left(\frac{dR}{ds}\right) = 2\frac{a^2\beta}{r^3}\sin v. \tag{19}$$

The equation which determines the value of δv, is [5367], namely,

$$d\delta v = 3\frac{dt^2}{r^2 dv}\int \mathrm{d}R + 2\frac{dt^2}{r^2 dv}r\left(\frac{dR}{dr}\right); \tag{20}$$

and the value of dR is

$$\mathrm{d}R = \left(\frac{dR}{dr}\right)dr + \left(\frac{dR}{dv}\right)dv + \left(\frac{dR}{ds}\right)ds. \tag{21}$$

The value of s is given by the equation [5376], namely,

$$s = \gamma\sin(gv - \Omega), \tag{22}$$

in which I have changed θ to Ω, in order to avoid confusion of symbols. Now (22) gives by differentiation

$$ds = g\gamma\,dv\cos(gv - \Omega). \tag{23}$$

If we neglect the eccentricity of the orbit we shall have $dr=0$, consequently the term $\left(\frac{dR}{dr}\right)dr$ will vanish from the value of dR. Now substituting the value of s, (22) in (18), and multiplying (19) by ds, which is given by (23), we shall get,

$$\left(\frac{dR}{dv}\right)dv = a^2\frac{\beta}{r^3}\gamma\,dv\,\{\sin(gv+v-\Omega) + \sin(gv-v-\Omega)\}. \tag{24}$$

$$\left(\frac{dR}{ds}\right)ds = a^2 g\frac{\beta}{r^3}\gamma\,dv\,\{\sin(gv+v-\Omega) - \sin(gv-v-\Omega)\}. \tag{25}$$

If we substitute these values in equation (21), and retain only the term depending on the angle $(gv - v - \Omega)$, it will become

$$dR = -a^2\frac{\beta}{r^2}\gamma\,(g-1)\,dv\sin(gv - v - \Omega.) \tag{26}$$

This gives by integration

$$\int dR = a^2\frac{\beta}{r^3}\gamma\cos(gv - v - \Omega). \tag{27}$$

If we substitute the value of s in equation (17) and multiply by r it will become

$$r\left(\frac{dR}{dr}\right) = -\ 3a^2\frac{\beta}{r^3}\gamma\cos(gv - v - \Omega). \tag{28}$$

Substituting (27) and (28) in (20), it becomes

$$d\delta v = -\ 3a^2\frac{\beta}{r^2 dv}\frac{\gamma}{r^3}\cos(gv - v - \Omega). \tag{29}$$

This is the same as LaPlace has given in [5368].

But LaPlace has given a second term depending on the same argument. This second term arises from the variation of the sun's disturbing force which is due to the variation of the moon's latitude produced by the earth's oblateness. The expression of this force is given in equation [5372] and is as follows:

$$\delta R = \tfrac{3}{2}m'u'^3r^2 s\delta s. \tag{30}$$

I shall now show that this value of δR is the same as the value of R given by equation [5362], except that it has a contrary sign.

According to [5374] we have

$$\tfrac{3}{2}m'u'^3r^2 = \tfrac{3}{2}\frac{m^2}{r} = 2\frac{g-1}{r}, \tag{31}$$

and if we substitute this in (30) or [5372] it becomes

$$\delta R = 2\frac{g-1}{r}s\delta s. \tag{32}$$

And if we substitute in this, the value of δs given by [5376], which reduced to the notation of this article is

$$\delta s = -a^2\frac{\beta}{(g-1)\,r^2}\sin v, \tag{33}$$

it becomes

$$\delta R = -\ 2a^2\frac{\beta}{r^3}\sin v. \tag{34}$$

This is the same as (16) or [5362] except that it has a contrary sign. This force is therefore the reaction of the force expended by the sun in giving motion to the moon's node, which in turn produces the inequality in the moon's latitude.

But in this second part of his work LaPlace seems to have committed a grave oversight, for he has treated his equation [5372] in the construction of [5373], as though δs were constant; whereas it is a function of both r and v, according to

[5376] which he afterwards uses in his reductions. However, as I have shown above that equation [5372] is the same as [5362], and has a contrary sign, it is unnecessary to pursue this part of the inquiry further, since it is evident that the whole value of δv must be derived from the value of R in [5362].

LaPlace has given the complete value of $d\delta v$ corresponding to the plane of the orbit, in [5379]; and he gives a correction in [5385] to reduce it to the plane of the ecliptic. It is apparent, however, that this correction does not exist, for LaPlace has shown in [923′] etc., where this subject is first investigated, that this correction is of the order of the square of the disturbing force; and as terms of that order have not been considered, it is evident that the value of that correction which he has given in [5385] is erroneous.

To complete this subject, it now remains to be shown that the value of R in equation [5362] gives the value of $d\delta v$ twice as great as LaPlace has found in [5368]. For this purpose I would remark that the value of $d\delta v$ given by means of [5367], is the correction to the *disturbed mean longitude*, and not to the undisturbed mean longitude. In order to correct for this condition it is necessary to add the term $3\frac{dt^2}{r^2dv}\int \mathrm{d}R$ to the first member, and this cancels the same term in the second member, thus leaving the correction to the *undisturbed mean longitude*, or δv equal to

$$\int 2\frac{dt^2}{r^2dv}\, r\left(\frac{dR}{dr}\right).$$

This will be apparent from the considerations given in § 54 of Book II of *Mécanique Céleste*, from which it appears that the function $\mathrm{d}R$ has a term of the form $\sin(at+\beta)$, in which a is very small, and gives by a double integration a^2 as a divisor; and LaPlace has shown in [1070′] that for this case we must increase the mean longitude by the quantity $3\frac{a}{\mu}\int ndt\int \mathrm{d}R$, which is equal to $3\int\frac{dt^2}{r^2dv}\int \mathrm{d}R$, or to the first term of the second member of equation [5367]. It therefore follows that the complete value of $d\delta v$ will be given by the equation

$$d\delta v = 2\frac{dt^2}{r^2dv}\, r\left(\frac{dR}{dr}\right). \tag{35}$$

and if we substitute the value of $r\left(\frac{dR}{dr}\right)$ given by [5365] it becomes equal to twice equation [5368], or identically the same as I have obtained by an entirely different method.

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