ART. V.—*The Law of Elastic Lengthening;** by J. O. THOMPSON, Haverford College.

IN the theory of elasticity the assumption is usually made that the lengthening of an elastic rod or wire is directly proportional to the stretching weight. Moreover experimental results have apparently justified this assumption in all cases where the lengthening was within the so-called limit of elasticity. Hooke† was the first to formulate this as a law, and his assertion "*ut tensio sic vis*," although occasionally questioned seems to have stood the test of experiment up to the present time. Wertheim‡ in his well-known investigations concerning the elasticity of the metals, and later Edlund§ in his measure-

* Translation made by the author for this Journal of a paper which appeared in Wiedemann's Annalen, vol. xliv, No. 11, 1891.

† R. Hooke, Lectures and Collections, London, 1679.

‡ Wertheim, Pogg. Ann., Ergänz, ii, (1848), p. 1.

§ Edlund, Pogg. Ann., cxiv, (1861), p. 1, and cxxvi, (1865), p. 539.

ments of the thermal effects produced by the stretching of wires find no inaccuracy in the law. Neither did Morin,* although the distance between the upper and lower marks on his wires amounted to 21 m.

Miller† too endeavors to prove that the elastic lengthening is proportional to the force. As far as I am aware, the old law has generally been regarded as axiomatic, and no one has ever made the attempt to replace it by a new one.

At the time of some experiments here made by Mr. Stradling this subject came up for discussion, and Prof. F. Kohlrausch invited me to investigate it in a series of careful experiments. In this place I desire to express my sincerest thanks to my honored teacher, Prof. Kohlrausch, for his advice and help in the course of these investigations.

In the first place it should be noticed that there is no good *a priori* reason why Hooke's law should be strictly true. When one examines the case closely it seems highly probable that deviations from the law should appear. When a wire is subjected to a tension the relative positions and distances of its molecules are altered, its thermal state is changed, and it becomes in these respects a new body. Accordingly that there should be in one and the same body a definite relation between the modulus of elasticity and the amount of deformation ought to occasion no surprise.

Since the experiments of others have amply proved that deviations from the old law of lengthening, if any really exist, must be very small, it is evident at the outset that the measurements, in order to detect these deviations, must possess a high degree of accuracy. The following experimental conditions must accordingly be secured: The wire must be long and free from all curvature. The temperature of the place where the experiments are made must remain approximately constant during an experiment. Further, account must be taken of several secondary phenomena the most important of which are the thermal effects due to the changing volume of the wire, and the elastic after-effect. The influence of these secondary phenomena will later be separately discussed.

The wires made especially for these experiments by Herr Obermaier of Nürnberg, gave me the first of the above mentioned conditions. The tower of the Physical Institute of the University of Strassburg, permitted the use of wires of the requisite length; and, since it was situated on the north side of the building, it possessed such a uniform temperature that

*Morin, Comptes Rendus, liv, (1862), p. 235.

† Miller, Aus d. Sitzungsber. d. math-phys. Classe der k. bayer. Akd. d. Wiss, 1882, Heft 4.

the total change in an entire forenoon amounted usually to less than half a Celsius degree. The length of the wires used was about 23 m.

In order to determine the mean temperature of the tower at any time, as well as the changes in the temperature, I used at first six thermometers hung at regular intervals in the tower. Afterwards at the suggestion of Prof. Kohlrausch I used in place of the six thermometers a fine brass wire 23 m. long, bearing a constant weight, and running parallel with the main-wire, the wire on which the measurements were made.

Since this thermometer-wire was only 9 cm. distant from the main-wire, and both were equidistant from cathetometer I, I was able, without changing the focus of the telescope, to observe the mark on the thermometer-wire and to determine the temperature accurately to the fortieth of a degree.

Description of Apparatus.

In the tower of the Institute are four solid brick piers, three of which terminate within the tower near the top. A large square beam 25 cm. thick was laid across two diagonally opposite piers, and this furnished an inflexible support for the upper end of the wire. A smaller beam was laid parallel with the larger, and on this was mounted a microscope focused on the upper mark on the wire. The larger beam rested on iron feet which were so adjusted that the whole arrangement stood absolutely firm. A movable cross-wire in the eye-piece of the microscope enabled one to measure a displacement of the mark of 0·005 mm. Since a weight of 70 kg. on the middle of the beam produced a sinking of only 0·03 mm. and since the heaviest weight used in any of my experiments was 18 kg. we may regard the upper mark as constant. Still to avoid all uncertainty the position of the mark was often noted and found to be unchanged.

At the lower end of the wire the following arrangement was used: Two cathetometers were mounted on the same stone block, one for the purpose of noting the position of the mark when no weight was on the pan, the other to note the exact point to which the mark was brought by any stretching weight. In order that a measurement might be made quickly, before the after-effect had time to make itself apparent, a damper immersed in glycerine was used. In this way one was enabled to make an accurate setting of the cathetometer within 10 sec. after the weight had been placed on the pan. To render impossible any friction between the damper and the sides of the surrounding vessel, the damper was fastend to the upper end of a light rectangular frame, to the lower end of which the pan was attached. The frame thus surrounded the vessel without touching it or the board on which it stood. Without some such arrangement there would have been friction between the damper and the sides of the vessel whenever the weight failed to be exactly in the centre of the pan. Furthermore in order to check quickly all motion the wire was passed through a fine hole in a stiff piece of leather, and the side of the frame playing between the prongs of a fork was prevented from swinging through a large arc. The weights used had been carefully calibrated so that the average inaccuracy was less than $\frac{1}{20000}$.

At first I used as lower mark a fine diamond-mark on the wire itself or on a glass bead. Later I used instead a very fine arrow-point printed on paper, and this secured an accuracy of 0·005 mm. in the setting of the cathetometer. Since in some cases the observed stretching of the wire amounted to more than 66 mm. it is apparent that when one, besides all this, takes mean values of many series of measurements, it is possible to attain a high degree of accuracy.

Thermal Effects within the Wire.

It was easy to foresee that the chief difficulties in measuring the true elastic lengthening would be those arising from the elastic after-effect and from the thermal effects due to changes in the volume of the wire. These thermal effects according to theory and observation are proportional to the stretching weight, provided that the stretching is not carried beyond the so-called limit of elasticity. By means of special observations with a thermo-pile I was able to prove that in all my experiments there was no perceptible deviation from this proportionality, and this fact may be taken as evidence that in the experiments the stretching was not carried beyond the proper limit. These thermal effects were in every case of greater influence than the after-effect.

The formula given by Sir Wm. Thomson for the diminution in temperature $\varDelta t$ caused by the stretching weight $\varDelta p$ is

$$\varDelta t = \frac{\mathrm{AT}\alpha}{wc} \varDelta p$$

where A is the heat equivalent of the unit of work, T the absolute temperature, α the coefficient of linear thermal expansion, w the mass of the unit of length of the wire, and c the specific heat of the metal by constant pressure.

Clausius* derives the same equation and states that Joule has corroborated it experimentally. Still the investigation of Joule on this point aimed at and attained merely a rough proof

* Clausius,' Mech. Wärmetheorie, 3 Aufl. I, p. 199, 1887.

of the above equation. Edlund* who later repeated these investigations with the best methods then known obtained thermal effects which were some 40 per cent smaller than the theoretical. Taking Edlund's results as a basis of calculation I found that in some cases these thermal effects could affect my measurements by as much as ·1 mm. The influence of the after-effect was much less, as will be shown later.

On account of the uncertainty in determining at any given instant the exact temperature of the wire after it had been stretched, I used in nearly all my experiments fine wires of from 0·2 mm. to 0·3 mm. diameter, for such wires assume quickly the temperature of the surrounding air. In order to estimate the rapidity with which the wire returned to its normal temperature after a weight had been placed upon the pan, I used a thermo-pile with mirror-galvanometer and scale. The thermo-pile consisted of two very fine wires, one of iron and the other of german-silver, fastened to the main-wire with the least possible amount of solder.

The sensitiveness as determined by the mean result of four experiments was 73 scale-divisions for 1° Celsius. In every case only the first deflection was recorded. The time of vibration of the needle was 6·5 sec. In determining the sensitiveness of the thermo-pile the junctions of the iron and german-silver wires with the galvanometer wires were immersed in petroleum of constant temperature, while the junctions with the main-wire, which were some 15 mm. apart, were immersed in a little rubber vessel containing distilled water. The temperatures of the water and of the petroleum were accurately determined, the circuit closed, and the deflection of the needle noted.

In measuring the thermal effects produced by the stretching of the wires the junctions of the thermo-pile wires with the main-wire were protected against air currents by a little cotton. Without this cotton the rapidity with which the wire near the junctions reassumed its normal temperature would have been even greater than it was. In a series of 8 consecutive determinations the mean position of rest of the galvanometer-needle was 635·7, the cooling of the brass wire occasioned by a weight of 1·5 kg. caused an average first deflection of 9·1 scale divisions toward the smaller numbers, and after 13 sec. or two vibrations the mean position of rest was 635·9.

A further proof of the rapid disappearance of the thermal effects is given by this very deflection of 9·1 scale-divisions. The first deflection was, as already mentioned, 73 scale-divisions for 1°. The cooling of the wire produced by the weight

* Edlund, Pogg. Ann., cxxvi, p. 557, 1865.

of 1·5 kg. amounted to 0·225° according to Edlund, and this difference of temperature, if it had remained constant during the time of the first deflection, would have produced a deflection of 16·4 scale-divisions instead of the actual 9·1. From this one can draw conclusions as to the rapidity of the disappearance of the thermal effects.

In all my experiments, even in extreme cases, I uniformly found that after a weight had been placed on the pan the needle would be deflected so as to indicate a certain cooling and then return with scarcely perceptible delay to its former position. Thus it was clear that after 13 sec. no visible trace of the cooling effect remained. If a difference of temperature of $\frac{1}{80}$°, enough to produce a contraction of 0·005 mm. in the length of the wire, had remained, the galvanometer would surely have indicated it.

The indications of the galvanometer were confirmed by noting the gradual contraction or lengthening of the wire immediately after unloading or loading the scale pan. In the first 10 or 12 sec. there would be a change of say 0·07 mm. in the length of the wire, and then in the following minute a further change of 0·01 mm. The first change one would naturally attribute chiefly to the thermal changes, the second to the after-effect. Since it is hardly possible in less than 12 sec. to place a weight on the pan and set the cathetometer with accuracy, it was to me a matter of indifference how much quicker the wire returned to its normal temperature.

Here it may be mentioned that Wertheim did not make his measurements until 5 or 10 min. after loading the pan. He waited, as he says, until the position of the mark became constant. In other words he did exactly what he should have avoided, he allowed his measurements to be affected by the most active part of the after-effect. Miller on the contrary went too far in the opposite direction and made measurements 2 sec. after stretching the wire. His results are obviously affected by temperature errors which he in a later article* seeks to correct.

The After-effect.

The complete calculation of the influence of the after-effect on my results would have required a large amount of time and trouble.

In the first place these phenomena have been studied chiefly in the cases of torsion and bending, and the laws of the after-effect in the case of stretching, at least as far as the metals are concerned, are unknown. Further in the present case we have

* Miller, Wied. Ann., xx, p. 94, 1883.

to deal with the superposition of different after-effects, and the problem becomes much more complex. The after-effect in the case of a strip of india-rubber is approximately proportional to the magnitude of the stretching.* Probably this applies to the metals as well, and if this is the case, the after-effect becomes of no account in the question we are discussing, provided the measurements are made always a definite time after the stretching of the wire. Moreover, since the after-effect in my experiments was invariably small I shall introduce in my results no correction for that factor.

Whenever the heaviest weights were used I set the cathetometer 13 sec. after placing the weight on the pan. This could be done by setting the cathetometer approximately beforehand. Careful measurements showed that this precaution was necessary only when the heaviest weights were used. In other cases it made no apparent difference whether the reading was made at the end of 13 sec. or a few seconds later. The measurements were made with rare exceptions at intervals of two minutes. In this time the after-effect following the release of the wire had almost entirely disappeared, as the results of two series of experiments given on pages 40 and 42 will show.

Still I wished to know exactly how soon this after-effect disappeared, and accordingly used the following plan: A fine flexible linen thread was tied to the bottom of the pan and then passed through a pulley beneath which was fastened to the floor. The thread was then made fast to the axle of a wheel, and thus by turning the wheel I could impart to the wire any desired tension.

In order to compare the influence of the after-effect with that of the thermal changes I proceeded as follows: By means of the thread a tension corresponding to a certain weight was imparted to the wire. The wire was kept stretched for about 13 sec. and then, by turning the wheel, released. During this process it was not necessary to remove my eye from the telescope. The wire came to rest without vibration in about one second, and a second later I had the telescope sighted on the mark. Then by means of the micrometer screw on cathetometer II, I measured the change in position of the mark in the following 11 sec., then the change from the 13th to the 30th second. By this time the mark had come back to its previous position. The results of the measurements with the brass wire are given in the following table:

* F. Kohlrausch, Pogg. Ann., clviii, p. 360, 1876.

Preceding Tension. kg.	Change from 2–13 sec. mm.	From 13–30. mm.	Total change. mm.
0·6	0·035	0·001	0·036
1·2	0·062	0·010	0·072
1·8	0·100	0·020	0·120

The figures in each column are mean values of 10, 20 and 8 measurements respectively. That the changes given in the second column come mostly from the thermal effects within the wire itself and not from the after-effect can be proved in the following way: If we insert in the equation which we have already mentioned in place of A the value $\frac{1}{682\cdot7}$ as obtained by Edlund we get

$$\varDelta t = \frac{1}{682\cdot7} \frac{T\alpha}{w\,c} \varDelta p.$$

In these experiments $T=285°$, $\alpha=0\cdot000018$, $w=0\cdot000528$ kg. per m., and $c=0\cdot094$. Consequently according to the equation $\varDelta t=0°\cdot15$ when $\varDelta p=1$ kg. Since the wire was 22·7 m. long the change in its length for 1° was 0·409 mm., or 0·0614 mm. when $\varDelta t$ was 0°·15. Therefore on removing stretching weights of 0·6, 1·2 and 1·8 kg. the lengthenings of the wire caused by the evolution of heat in the first instant should be 0·037, 0·074 and 0·111 mm. respectively. These figures approximate those in the second column of the above table as closely as we could expect in an experiment of this kind.

From what has been said it is safe to conclude that the necessary correction on account of the after-effect is very small. The stretching of the wire caused by a weight of 1·2 kg. was 43·55 mm., and the change of 0·01 mm. which took place after the first 13 sec. is insignificant in comparison.

Method of Measurement.

The method followed in making all measurements was the same. After noting the position of the mark on the thermometer-wire the telescopes of both cathetometers were sighted on the mark on the main-wire, and careful readings were made. Then a weight of 0·2 kg. was put on the pan, the consequent lowering of the mark was quickly measured by cathetometer I, and then the weight was immediately removed.

Two minutes later any shifting of the zero-point was measured by means of the micrometer screw of cathetometer II, and then the same process was repeated with a weight of 0·4 kg. In some cases the weight was increased to as much as 1·8 kg.

By means of the micrometer screw of cathetometer II. I could measure the shifting of the zero-point accurately to 0·005 mm. In this direction only an extremely small inaccuracy was possible because almost without exception the gradual lowering of the zero-point, in consequence of rising temperature and other causes, allowed the screw to be turned always in the same direction.

At the end of each series of observations the zero-point was again noted by both cathetometers, and the displacement measured. The difference in the results given by the two instruments was on an average less than 0·005 mm. Finally, in order to eliminate the influence of any change of temperature in the tower, the position of the mark on the thermometer-wire was again noted.

Experiments with Brass Wire.

As an example of the process just described I give the following series of measurements:

Apr. 25, 10h. 5m. Thermometer-wire, 540·71
Temperature, 9°·5

Added weight. kg.	Time. h. m.	Cath. I. mm.	Cath. II. mm.	Lengthening. mm.
0	10 8	548·96		
0·2	9	41·84	0·15	7·12
0·4	11	34·70	"	14·26
0·6	13	27·46	"	21·50
0·8	15	20·18	"	28·78
1·0	17	12·81	0·14	36·14
1·2	19	5·39	"	43·56
1·4	21	497·885	"	51·065
1·6	23	90·26	0·135	58·685
1·8	25	82·60	0·125	66·335
·0	27	548·92	0·105	

10 h. 30 m. Thermometer-wire, 540·68

The lowering of the zero-point during these measurements was 0·04 mm. according to cathetometer I, and 0·045 mm. according to cathetometer II. The thermometer-wire, which was just like the main-wire, lengthened by 0·03 mm. during this time. The rise in temperature was accordingly $\frac{3}{40}$°, about twice the ordinary rise. The above table indicates, as already shown by the table on p. 39, that with an added weight of 1·8 kg. the after-effect begins to be noticeable. The lengthening produced by this weight I have given, but not taken into account in my later calculations. The initial load plus 1·8 kg. amounted to nearly half the breaking weight.

Table of 10 Series of Measurements.

Kg.											Mean mm.
0·2	7·10	·13	·12	·11	·105	·12	·11	·11	·12	·09	7·111
0·4	14·25	·265	·26	·28	·285	·27	·265	·27	·28	·27	14·269
0·6	21·47	·50	·50	·50	·485	·495	·49	·475	·49	·49	21·489
0·8	28·76	·775	·78	·77	·75	·785	·77	·77	·79	·77	28·772
1·0	36·11	·125	·14	·145	·115	·13	·13	·11	·12	·12	36·124
1·2	43·53	·57	·56	·55	·555	*	·57	·55	·545	·56	43·554
1·4	51·05	·07	·065	·07	·065	·06	·065	·09	·055	·08	51·067
1·6	58·66	·71	·685	·71	·66	·69	·675	·68	·69	·675	58·683
1·8	66·33	·355	·335	·36	·32	·35	·31	·32	·34	·32	66·334

Ten additional series of measurements were made. The mean values derived from the 20 series are given in the following table under *x observed.*

p.	*x* observed.	*x* calculated.	Observed-calc.
0·2 kg.	7·111	·110	+0·001
0·4	14·272	·271	+ 1
0·6	21·488	·488	± 0
0·8	28·770	·770	± 0
1·0	36·119	·122	− 3
1·2	43·554	·554	± 0
1·4	51·076	·071	+ 5
1·6	58·679	·681	− 2
1·8	66·341	----	----

Mean temperature, 9°.
Cross-section of wire, 0·0627 mm.^{2}
Length of wire, 22700 mm.
Specific gravity of wire, 8·42
Initial load, 0·665 kg.
$p =$ added weight, and $x =$ lengthening.

Expressing the result of the measurements by an equation of the form

$$x = ap + bp^2 + cp^3$$

I obtained the equation

$$x = 35{\cdot}4385\, p + 0{\cdot}5353\, p^2 + 0{\cdot}1487\, p^3$$

The calculation of the most probable values of the constants a, b and c, was effected in this case, as in all the other cases, by means of the method of least squares. The mean error of the individual measurements calculated from the 10 lengthenings produced by a weight of 1·4 kg. and given in the table on p. 41, is 0·011 mm. and the probable error of the mean result is 0·0024 mm. Two minutes after the weighting and release of the wire the mean lowering of the zero-point was 0·009 mm.

* No measurement was made. If I had waited for the mark to come to rest the after-effect would have been noticeable.

The initial load, consisting of frame, scale-pan, damper, and half the weight of the wire itself, amounted to ·665 kg. This was probably twice as much as was really necessary to keep the wire straight. In later measurements a lighter frame was used.

Experiments with Copper Wire.

The following is one of ten series of measurements made May 23rd.

Time 4 h. 7 m. Thermometer-wire 540·70

Added weight. kg.	Time. h. m.	Cath. I. mm.	Cath. II. mm.	Lengthening. mm.
0	4 9	554·82		
0·2	10	49·29	0·15	5·53
0·4	12	43·73	"	11·09
0·6	14	38·14	"	16·68
0·8	16	32·51	"	22·31
1·0	18	26·84	0·14	27·97
1·2	20	21·14	0·13	33·66
0	22	554·795	0·12	

Lowering of zero-point, 0·025 0·03
4 h. 27 m. Thermometer-wire 540·70

At 4 h. 33 m., the beginning of the next series of observations, the zero-point had risen 0·015 mm. on account of the after-effect. This was generally the case between any two series. The influence of the after-effect, although rather more marked here than in the case of the other metals experimented upon, was not enough to affect the value of the measurements.

On this day the temperature was remarkably constant. The total steady lengthening of the thermometer-wire between 2 h. 50 m. and 5 h. 23 m. amounted to only 0·02 mm.

The following table gives the results of eight successive measurements made with the copper wire May 23rd.

p. kg.									Mean mm.
0·2	5·535	·535	·54	·53	·53	·53	·53	·535	5·533
0·4	11·085	·08	·10	·085	·09	·09	·09	·095	11·090
0·6	16·675	·675	·685	·685	·685	·68	·67	·68	16·680
0·8	22·31	·305	·31	·31	·30	·31	·31	·31	22·308
1·0	27·97	·94	·98	·985	·95	·97	·94	·945	27·960
1·2	33·665	·65	·72	·685	·67	·66	·64	·62	33·664

The following table gives the mean values derived from 16 series of measurements with copper wire.

p. kg.	*x* observed. mm.	*x* calculated. mm.	Observed-calc. mm.
0·2	5·531	·529	+ 0·002
0·4	11·084	·086	− 2
0·6	16·671	·673	− 2
0·8	22·298	·294	+ 4
1·0	27·949	·951	− 2
1·2	33·646	·646	± 0

Mean temperature, 13°·5
Cross-section of wire, 0·0641 mm.^{2}
Length of wire, 22690 mm.
Specific gravity of wire, 8·99
Initial load 0·192 kg.

The figures of the third column were calculated according to the equation

$$x = 27{\cdot}578\, p + 0{\cdot}3193\, p^2 + 0{\cdot}0538\, p^3.$$

The initial load was 0·192 kg. Measurements showed that 0·15 kg. sufficed to hold the wire straight, while 0·10 kg. was insufficient.

Experiments with Steel Wire.

The following table gives the mean result of 20 series of measurements.

p. kg.	*x* observed. mm.	*x* calculated. mm.	Observed-calc. mm.
0·2	7·078	·077	+ 0·001
0·4	14·196	·197	− 1
0·6	21·358	·358	± 0
0·8	28·558	·558	± 0
1·0	35·792	·793	− 1

Mean temperature, 13°
Cross-section of wire, 0·03263 mm.^{2}
Length of wire, 22700 mm.
Specific gravity of wire, 7·74
Initial load, 0·491 kg.

The figures in the third column were calculated according to the equation

$$x = 35{\cdot}2725\, p + 0{\cdot}5725\, p^2 - 0{\cdot}0525\, p^3$$

An evidence of the insignificance of the after-effect may be seen in the fact that in ten successive series of measurements the lengthening caused by the maximum load deviated in no case more than ·0225 mm. from the mean. Two minutes after the release of the wire the mean position of the zero-point was ·01 mm. lower than immediately before this maximum load was put on the pan.

Experiments with Silver Wire.

The following table gives the mean result of eight series of measurements.

p. kg.	x observed. mm.	x calculated. mm.	Observed-calc. mm.
0·2	7·898	·896	+ 0·002
0·4	15·820	·822	− 2
0·6	23·775	·776	− 1
0·8	31·758	·756	+ 2
1·0	39·762	·762	± 0

Mean temperature, 14°
Cross-section of wire, 0·0687 mm.²
Length of wire, 22690 mm.
Specific gravity of wire, 10·00
Initial load, 0·593 kg.

The figures in the third column were calculated according to the equation

$$x = 39{\cdot}4030\, p + 0{\cdot}3905\, p^2 - 0{\cdot}0313\, p^3$$

This wire was not so free from curves as the others, and consequently it was necessary to begin with a comparatively heavy initial load.

The influence of the after-effect was as a rule so slight that it could not be measured. After each release of the wire the mark returned quickly to the zero-point, and the sinking of the zero-point was explained almost entirely by the gradual rise in temperature in the tower.

Calculation of the True Modulus of Elasticity.

From the results given above it follows that the modulus of elasticity is not a constant, but in every case a function of the tension. From the equation

$$x = ap + bp^2 + cp^3$$

which gives the relation between the elastic lengthening and the stretching weight we obtain

$$\frac{dx}{dp} = a + 2bp + 3cp^2$$

and with the help of this latter equation we can compute the modulus for any tension whatever.

The almost perfect agreement between the observed and calculated values of x makes it extremely probable that, inasmuch as the initial load was slight, we may extend the application of this equation backwards to an initial load of zero.

Thus we shall be in a position to determine the true modulus of elasticity, the modulus of the body before it has been subjected to any deformation whatever.

In order to determine this true modulus we may proceed as follows:

Let X_0 be the lengthening caused by the initial load P_0*
" X " " " " total weight $P^\circ+p$

Then the observed lengthening $x=X-X_0$

$$\text{Let } X=\alpha P+\beta P^2+\gamma P^3 \tag{1}$$
$$\text{and } X_0=\alpha P_0+\beta P_0{}^2+\gamma P_0{}^3 \tag{2}$$

Equation (2) subtracted from (1) gives

$$x=(-\alpha P_0-\beta P_0{}^2-\gamma P_0{}^3)+\alpha P+\beta P^2+\gamma P^3 \tag{3}$$

The results of the measurements were given in the following form

$$x=a(P-P_0)+b(P-P_0)^2+c(P-P_0)^3$$
$$=(-aP_0+bP_0{}^2-cP_0{}^3)+(a-2bP_0+3cP_0{}^2)P+(b-3cP_0)P^2+cP^3 \tag{4}$$

Equating the coefficients of like powers of the variable P in the two expressions for x in (3) and (4) we obtain

$$\alpha=a-2bP_0+3cP_0{}^2$$
$$\beta=b-3cP_0$$
$$\gamma=c$$

Equation (1) which gives the relation between elastic lengthening and stretching weight when one begins with an initial load zero becomes in the case of the steel wire for instance

$$X=34{\cdot}672P+0{\cdot}6498P^2-0{\cdot}0525P^3$$

$$\text{and} \qquad \left(\frac{dX}{dP}\right)_{P=0}=34{\cdot}672$$

With an infinitesimal stretching weight the formula for the modulus of elasticity is

$$E=\frac{l}{q}\cdot\frac{dP}{dX}$$

where l is the length and q the cross-section of the wire. Therefore in this case

$$E=\frac{22683}{0{\cdot}03263}\cdot\frac{1}{34{\cdot}672}=20050$$

The length of the unstretched wire l can be found with the help of equation (2).

*P_0 represents the weight of the frame with pan and damper, increased by half the weight of the wire.

The equations giving the relation between elastic lengthening and stretching weight when one begins with zero load are as follows:

Steel.......... $X=34{\cdot}672P+0{\cdot}6498P^2-0{\cdot}0525P^3$
Brass .... $X=34{\cdot}924P+0{\cdot}2386P^2+0{\cdot}1487P^3$
Silver .... $X=38{\cdot}907P+0{\cdot}4462P^2-0{\cdot}0313P^3$
Copper .... $X=27{\cdot}461P+0{\cdot}2883P^2+0{\cdot}0538P^3$

The specific gravities, cross-sections, and moduli calculated by the above method are given in the following table.

	Sp. Gr.	Cross-section.	True Modulus.	Modulus I.	Modulus II.
Steel	7·74	0·03263	20050	19430	19230
Brass ...	8·42	0·0627	10370	9820	9450
Silver ...	10·00	0·0687	8490	8300	8250
Copper ..	8·99	0·0641	12890	12620	12420

In order to show still more clearly the dependence of the modulus on the tension I add two more columns. The next to the last column gives the modulus which one would obtain if only the lengthening caused by the maximum added load were taken into account. The last column, the modulus which would be obtained if only the increase of lengthening caused by the last 0·2 kg. should be made the basis of calculation. As will be seen, the true modulus of elasticity of the brass wire is nearly 10 per cent greater than the one given in the last column.

Influence of Contraction of Cross-Section.

When a wire is stretched so that each unit of length increases by δ, the cross-section becomes $Q_0(1-2\mu\delta)$ if Q_0 is the original cross-section. Theoretically μ can have in different bodies any value between the limits 0 and $\frac{1}{2}$. Supposing that in the case of steel $\mu=0{\cdot}294$, as Kirchhoff states, we find that for a weight of 1 kg. $\delta=35{\cdot}8\div22700=0{\cdot}0016$ and the cross-section is accordingly $Q_0(1-0{\cdot}00094)$. If we wished to refer the modulus of elasticity in this case to this diminished cross-section, the coefficients of our equations would suffer a perceptible change. But it is not customary to do this, and we shall refer the modulus to the condition of the unstretched wire. Further it should be noticed that the value of μ is not definitely known; in fact, there is ground for believing that in one and the same body it is a variable quantity.

Larger Wires.

In order to be perfectly sure that the phenomena which I have described were not confined to fine wires, I made careful measurements with larger wires. The loads placed on these were gradually increased to a maximum of 18 kg. and without exception the results obtained were similar to those which I have reported. The reasons however why I preferred to use fine wires are first, because in these the thermal effects vanish more rapidly, and second, because the loading and unloading can be done in shorter time, and thus the after-effect is more completely eliminated.

Earlier Investigators.

Since these investigations have yielded results which are at variance with those hitherto obtained, it is desirable to search for an explanation of the discrepancy. It is evident that the presence of curves and bends in the wire would give to $\frac{x}{p}$ too large a value which would gradually decrease as the wire grew straight. This circumstance would account for Stradling's remark* according to which the modulus of elasticity would increase with the tension, and is the probable explanation of the fact that Wertheim† in the case of annealed platinum finds the modulus increasing with the weight. Further the almost universal result obtained by other investigators, namely that $\frac{x}{p}$ has a constant value, can be explained in the very same way. The diminution of the modulus alone considered would make $\frac{x}{p}$ larger, while the gradual straightening of the wire would tend to make it smaller. The combination of these two factors could easily lead to the conclusion that within the limits of errors of observation $\frac{x}{p}$ has a constant value.

Discussion of Results.

Whatever the nature of elastic force may be, whether it is an essential property of matter, or a derived property which can be accounted for by the rotation of atoms, the fact was established by Wertheim that between the elasticity of a body and its density there exists an intimate relation. Conclusion I of his investigation is as follows:‡ "The coefficient of elasticity is not a constant for one and the same metal. All circumstances which increase the density make it larger, and conversely." These words justify a deduction which Wertheim

* Stradling, Wied. Ann., xli, p. 332, 1890.
† Wertheim, Pogg. Ann. Ergänz., ii, p. 46, 1848.
‡ Wertheim, l. c. p. 69.

himself seems not to have made, for conclusion VII shows that he did not look for any change in the modulus of elasticity until the density of the body had been permanently altered. Wertheim proved that the density of a wire before an experiment differs very slightly from its density after it has been broken by its load. He therefore concludes that in one and the same wire, even when it is in different conditions of equilibrium, the modulus of elasticity can vary only a very little.

But it should be kept in mind that, although on account of the contraction of cross-section the change in density is slight, the mean molecular distance in the direction in which the tension is exerted probably increases by a very considerable amount. And it is probable that the variability of the modulus of elasticity should be attributed not so much to the alteration in density of the body considered as a whole, as to this change of mean molecular distance in the direction of the tension. According to Wertheim's own assumption we should have in one and the same metal

$$\text{Mod. of elasticity} \times a^7 = \text{constant},$$

where a represents the mean molecular distance. According to this any change in a would produce in the modulus a change proportionally seven times as great, and this in the case of the stretching of a wire can become very noticeable. According to my own measurements to be sure a much higher power than the seventh must be assumed in the formula.

If it is universally true that an increase of the mean molecular distance causes a diminution of the modulus of elasticity according to a definite law, then in those metals which have the largest coefficients of thermal expansion the decrease of the modulus of elasticity with the temperature ought to be most rapid. That this is actually the case has been already mentioned by Miller* and Katzenelsohn.† A further deduction would be that since by higher temperatures the coefficient of thermal expansion grows larger, the modulus should here decrease more rapidly than by lower temperatures. That this is also true has been proved by the measurements of Kohlrausch and Loomis‡ and Pisati.§

Further if a substance like india rubber has a negative coefficient of thermal expansion, it would follow from our principle that as the temperature increases the modulus should increase too. Just such an increase in the modulus of elasticity

* Miller, Wied. Beibl., xi, p. 211, 1887.
† Katzenelsohn, Wied. Beibl., xii, p. 307, 1888.
‡ F. Kohlrausch and Loomis, Pogg. Ann., cxli, p. 498, 1870.
§ Pisati, Wied. Beibl., i, p. 305, 1877.

of india rubber has been observed by Miller* *These facts and the results of my experiments justify the conclusion that in every case the modulus of elasticity of a body is a function of the molecular distance, and that every agency, whether it be heat or mechanical force, which increases this molecular distance, produces a diminution of the modulus of elasticity.*

This relation which is enunciated here for the first time as a general law has, as far as I am aware, no exception which cannot be explained by the necessary errors of observation.

It is interesting to notice that in many instances the theory of probabilities is able to deduce from the measurements of Wertheim results similar to mine. The following values for the moduli of pure gold (Wertheim, l. c. p. 30), and pure silver (p. 45) show on their face an evident diminution as the stretching weight increases.

kg.	gold.	silver.
3	7030	
4	6391	
5	5021	7701
6	5492	7578
7	5340	6476
8	5291	7555
9	4972	7213
10	5140	7123

In other cases, for instance in the measurements made with copper (p. 35) and silver (p. 31) where consecutive values of the modulus differ by as much as 10 per cent it is necessary to apply the theory of probabilities.

In the previously mentioned measurements with annealed platinum the opposite tendency can probably be explained by curvature of the wire.

Whether in the method of flexure it is possible to detect a dependence of the modulus on the amount of the load I have not had time to investigate. Since however the compression on one side of the neutral layer is equal to the extension on the other, and consequently the density remains either exactly or very nearly the same, it is probable that in this case no change in the modulus of elasticity could be observed.

A number of relations and applications of this true law of elastic lengthening to other physical laws, and other phenomena observed in the course of my experiments, I shall discuss at some future time.

* Miller, l. c.

Conclusions.

I. The generally accepted law of elastic lengthening $x=aP$, according to which the lengthening x is proportional to the stretching weight P, is only an approximation.

II. The relation between elastic extension and stretching weight can be expressed by an equation of the following form:

$$X=\alpha P+\beta P^2+\gamma P^3$$

III. The modulus of elasticity of the undeformed body can be calculated with the help of the equation

$$\left(\frac{dX}{dP}\right)_{P=0}=\alpha$$

IV. The true moduli of elasticity calculated in this way may be as much as 10 per cent larger than those determined in the ordinary way. Consequently it will be necessary to recalculate physical constants which depend on the modulus of elasticity.

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