ART. LIII.—*Resonance Analysis of Alternating Currents*;*

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I. *Introduction.*

THE presence of upper harmonics in an alternating current wave is a fact which deserves careful consideration both on account of the purely scientific interest which is attached to it, and also on account of the technical bearing of electrical resonance upon the construction of conductors possessing appreciable distributed capacity.

That alternating current and electromotive force waves of a very great variety of forms can be produced by properly designing the pole-pieces of the field magnet, and the iron core of the armature of an alternator is a fact nearly as old as the discovery of electromagnetic induction. Fully as old is also the knowledge that a great variety of alternating current and electromotive force waves can be obtained by the induction of an intermittent current.

A careful investigation of these waves was first made more than forty years ago by Lenz† and Koosen,‡ who employed alternators with iron in the armature. They plotted these waves from the instantaneous values of current and electromotive force obtained by means of the now well-known *revolving sliding contact.* Employing the same method of investigation Joubert§ showed in 1880 that the electromotive force wave obtained from an eight pole Siemens alternator without iron in the armature is very nearly a sine wave. *The method is now known as Joubert's method of the sliding contact.* The name "*indicator diagram*" has been applied to the wave curves of current and electromotive force obtained by Joubert's method, and very properly, I think, because they do very clearly indicate the action of alternating current apparatus.

Our knowledge of the action of alternating current apparatus has been extended quite considerably by these indicator diagrams.

Although much must be said in favor of the sliding contact method of obtaining indicator diagrams, yet it must be also acknowledged that the method is a very laborious and uninteresting process of investigation. Many attempts have been made to devise some optical or some automatic method, but

* Read before the Annual Meeting of the American Institute of Electrical Engineers at Philadelphia, May 17th, 1894.

† Pogg. Ann. lxxvi, p. 494, 1849; xcii, p. 128, 1854.

‡ Ibid., lxxxvii, p. 386, 1852.

§ Comptes Rendus, vol. xci, p. 161, 1880; Ann. de l'école super. 10, p. 131, 1881.

with little success. There is another reason why a new method of studying alternating current waves seems desirable. It is this: the method of sliding contact is not sufficiently sensitive to detect small deviations from a true sine wave, and consequently it is not capable of following up the causes of these deviations, when the effects seem to be absent. For instance, the primary current of a transformer can differ very much from a true sine form when the secondary circuit is open, but when a large current is flowing through an approximately non-self-inductive secondary circuit, then the primary current can be made to differ inappreciably from a true sine wave. *The question arises now, what becomes of these causes when the secondary carries a heavy non-self-inductive load?*

This question is of deep scientific interest; it is also of considerable technical importance. For, if these causes are present at all loads and only hidden by the principal wave, then, considering that these hidden small causes can produce large effects when conditions favoring resonance arise, it is evident that they must be carefully watched and guarded against in the construction of lines possessing appreciable distributed capacity. I do not think that indicator diagrams obtained by the method of sliding contact are capable of giving a definite answer to this important question.

The method of analyzing alternating current waves by electrical resonance which I employed in the following investigation was suggested by me a year ago.* It is the object of this paper to describe this method at some length and to illustrate, by some of the more definite results so far obtained and relating principally to the causes which produce distortions in simple harmonic waves, the simplicity, sensitiveness, and reliability of the method. I shall also point out that this method of resonance analysis works quite satisfactorily even in those cases, alluded to above, where the sliding contact method would in all probability fail to detect any distortion whatever.

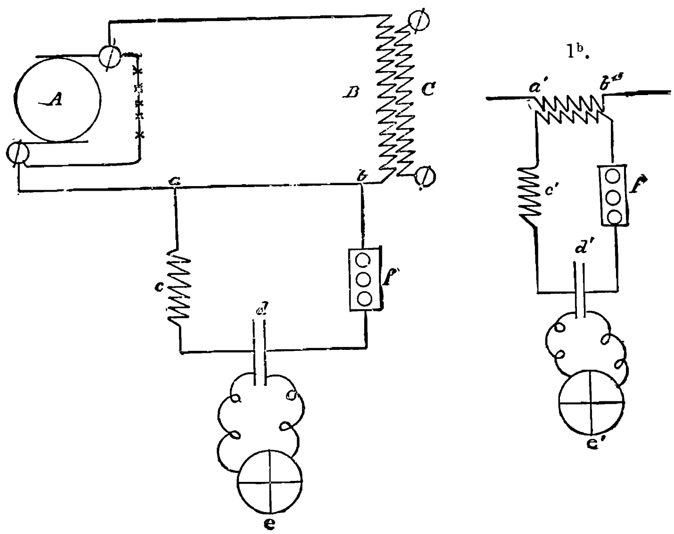
II. *Description of the Method.*

Consider the following arrangement of circuits:—The non-self-inductive resistance, *ab*, fig. 1^a, is inserted in the circuit of an alternator A and the primary B of a transformer. In shunt with *ab* is a circuit *acdb* consisting of an inertia coil *c* of large number of turns of copper wire of low resistance, about 10 ohms, but containing no iron, and a mica condenser *d* divided into subdivisions ranging from ·001 *M.F.* up. In shunt with the condenser *d* is an electrostatic voltmeter *e*. The self-induction of the coil *c* can be varied by throwing a larger or a smal-

* M. I. Pupin, "Electrical Oscillations of low frequency and their Resonance," this Journal, vol. xlv, p. 429, May, 1893.

ler number of its sections into the circuit. The resistance can be varied by a rheostat *f*. Suppose now that the self-induction of *c* is kept constant, and that the capacity of the condenser *d* is gradually increased from zero up. Whenever a capacity has been reached which with the self-induction of the circuit *acdfb*

1^a.



produces resonance with one of the harmonics in the main circuit then the resonant rise of potential will produce a large deflection in the voltmeter. In this manner all the harmonics which are present in the current of the main circuit can be detected in the course of a few minutes. If the resonator circuit *acdfb* is placed in shunt with the non-self-inductive circuit *g* (this circuit is represented in fig. 1^a by a line beaded with asterisks and running from one pole of the alternator to the other) consisting of a bank of incandescent lamps then the harmonics of the impressed electromotive force can be detected in the same manner. The ratio of the amplitudes of these harmonics to that of the fundamental can also be determined by this method, *if desirable*, provided the conditions of the experiment are properly arranged. For let the current in the main circuit be

$$x = a_1 \sin pt + a_3 \sin 3pt + \ldots\ldots + a_{2\alpha+1} \sin(2\alpha+1)pt + \ldots$$

then the drop between *a* and *b* can be represented by

$$e = b_1 \sin pt + \ldots\ldots + b_{2\alpha+1} \sin(2\alpha+1)pt + \ldots$$

$$\text{where } b_{2\alpha+1} = a_{2\alpha+1} r$$

and r = ohmic resistance between *a* and *b*. Denoting now by:

L the self-induction of the resonator *acdfba*
R the resistance " " "
C the capacity " " "

then it can be easily shown* that the current in the resonator will be:

$$y=\sum_{a=0}^{\infty}\frac{b_{2a+1}}{\sqrt{(2a+1)^2p^2\left\{\frac{1}{(2a+1)^2p^2C}+L\right\}^2+R^2}}\sin\left[(2a+1)pt+\varphi_{2a+1}\right]$$

If, therefore, the capacity C is adjusted in such a way that

$$\frac{1}{(2a+1)^2p^2C}-L=0$$

then the circiut will be in resonance with the harmonic of frequency $\frac{(2a+1)p}{2\pi}$; and if L is sufficiently large and R sufficiently small (two conditions which are very easily fulfilled) the current y will in general to within a small fraction of a per cent be given by

$$y=\frac{b_{2a+1}}{R}\sin(2a+1)pt$$

The amplitude of the potential difference in the condenser which is measured by the voltmeter *e* is then given by

$$P_{2a+1}=\frac{(2a+1)pL}{R}b_{2a+1}$$

In the same way we obtain for the fundamental frequency

$$P_1=\frac{pL}{R}b_1$$

Hence

$$\frac{P_{2a+1}}{P_1}=(2a+1)\frac{b_{2a+1}}{b_1}$$

This gives the ratio of the amplitude a_{2a+1} of the harmonic of frequency $\frac{(2a+1)p}{2\pi}$ to that of the fundamental. Let $a=2$, then,

$$\frac{1}{5}\frac{P_5}{P_1}=\frac{b_5}{b_1}=\frac{a_5}{a_1}$$

The voltmeter readings which give P_5 and P_1 magnify that ratio five times, in the case of the fifth harmonic, and it can be easily seen that a similar relation holds true for other harmonics. This is a very desirable feature of the method, considering that the amplitudes of the upper harmonics are generally small in comparison to the amplitude of the fundamental, especially when the secondary circuit of the transformer carries a load.

* For further information see author's paper cited above.

When quantitatively very accurate results are desired then a low resistance, say one ohm, should be used for the section *ab* and an electrometer capable of giving a large deflection for ten volts.

The principal interest, however, *in the study of the distortion of alternating current waves, is centered* not so much in the exact ratio of the amplitudes of these harmonics to the amplitude of the fundamental wave, as it is *in the causes producing these harmonics and the conditions which modify the effects of these causes.* Hence a quantitatively less accurate arrangement will do, provided that it is very sensitive, simple, and easily manageable. Such an arrangement is given, fig. 1[b].

It differs from that given in fig. 1[a] in the substitution of an air core transformer coil *a'b'* for the non-self-inductive resistance *ab*. The secondary of this coil forms a part of the resonator circuit. For every harmonic of the inducing current we shall have a harmonic electromotive force of the same frequency in the resonant circuit. By varying the capacity in the resonator and watching the voltmeter needle, we can tell by the deflection of the needle, whenever we have reached the capacity which with the self-induction of the resonator brings this circuit into resonance with one of the harmonics. A reference to fig. 2 will explain this more clearly.

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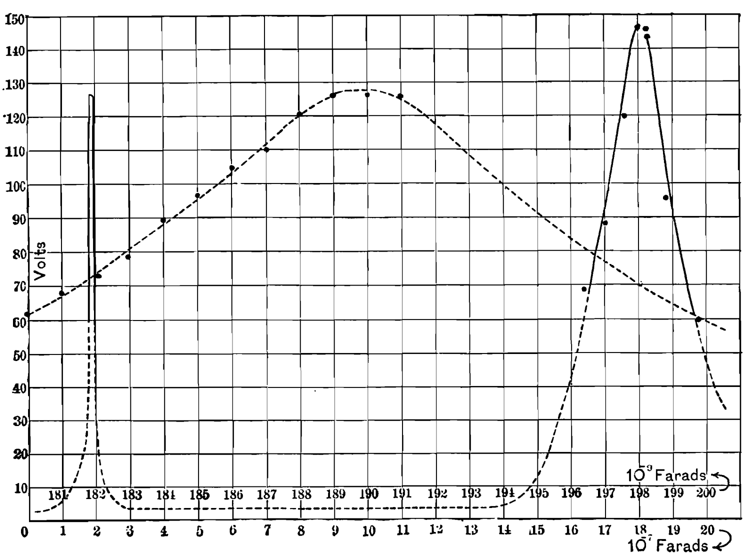


In this figure the lower horizontal row of figures refers to the two-peaked curve; the upper row refers to the dotted flat-peaked curve. The vertical row denotes the voltmeter readings in volts. Consider now the two-peaked curve. It expresses the law of variation of the voltmeter readings when the capacity of the resonator circuit is varied from 0 to 2 microfarads, the self-induction being kept constant. The readings are recorded in Table I.

TABLE I.

Capacity of the condenser in microfarads.	Voltmeter readings in volts.
·18	62
·181	68
·182	73·5
·183	79
·184	89
·185	96
·186	104
·187	110
·188	120
·189	126
·190	127
·191	125
·194	99
·198	71
·202	very low
1·65	69
1·70	89
1·75	120
1·80	146
1·808	146
1·817	145
1·897	96
1·976	60

The voltmeter employed in these experiments was a Sir William Thomson's multicellular voltmeter with a range from 60 to ·240 volts. The curve was obtained from a 10 H. P. Fort Wayne 8 pole alternator with a smooth core armature feeding a 5 K. W. Stanley transformer (closed magnetic circuit), the secondary circuit being open. It is seen that resonance took place at ·190 M. F. and 1·8 M. F. The capacity of the inertia coil c', fig. 1^b and of the voltmeter as gathered from all experimental data was about ·011 M. F., so that the real capacities at which resonance took place were ·201 M. F. and 1·81 M. F., that is in a ratio to each other as $1:3^2$. It will be seen, however, that a very accurate knowledge of capacity is not required in the experiments described in this paper.

The frequencies detected by the two-peaked curve, which I shall call the *resonance diagram*, were therefore the fundamental and the 1st odd harmonic, that is the harmonic of three times the frequency of the fundamental. The resonance diagram has, of course, as many peaks as there are harmonics in the inducing current.* The dotted curve (flat-peaked) in fig. 2 was plotted on an enlarged scale from the readings taken in detecting the first harmonic represented by the sharp peak of the resonance diagram, and represents this peak spread out, so as to show how the various readings fit into a well defined and symmetrical curve such as required by theory. It also shows that a condenser of small subdivisions should be employed in detecting higher harmonics.

III. *Description of Experiments.*

The *resonance diagram* obtained by the method of fig. 1^b gives the number of harmonics which are present in the inducing current. It does not give the exact value of the amplitudes of these harmonics. It would be somewhat premature to discuss the theory of the resonance diagram obtained by this arrangement and to show how the ratio of the amplitudes of the harmonics to that of the fundamental frequency in the inducing current, that is the exact color of this current, could be calculated from the ratio of the height of the peaks in the resonance diagram. Suffice it for the present to mention only that the peaks of this diagram represent the amplitudes of the harmonics magnified about proportionally to the square of the frequency. For instance, the resonance diagram of fig. 2 tells us that the amplitude of the 1st odd harmonic in the inducing current is about one-ninth of the amplitude of the fundamental. The determination of the exact value of this ratio was not the object of the following experiments. *Their aim was to detect the presence of harmonics, to trace their origin and to study their variation with the variation of the load, and of other variable elements of the circuit on which these harmonics seem to depend.*

Preliminary Tests.

In order to form an estimate in how far the experimental data obtained by the arrangement of fig. 1^b agreed with the theory the following tests were applied:

*I have never detected an even harmonic in alternating current waves produced by ordinary commercial alternating current apparatus, and conclude, therefore, that these harmonics do not exist in such cases. For asymmetrical machines this would obviously not hold true. Alternators with slotted armatures give waves in which all the odd harmonics up to the harmonic of nine times the frequency of the fundamental can be detected. As a rule the first odd harmonic is the strongest.

a. *Study of the damping effect of the dielectric in the condenser.*

Let L = self-induction of the resonator circuit.
R = resistance of the resonator circuit.
P = amplitude of the difference of potential in the condenser when point of resonance has been reached for a given frequency.
E = amplitude of impressed electromotive force in the resonant circuit.

then according to theory

$$P = \frac{pL}{R} E.$$

Hence if R alone is varied P will vary also but in such a way that

$$P R = \text{constant}.$$

That is to say if we vary the resistance of a resonant circuit and tabulate the voltmeter deflection for every particular resistance and then plot a curve taking the resistance for abscissae and the voltmeter readings for ordinates we should, according to theory, obtain an equilateral hyperbola. Curves II and III, fig. 3, were obtained in this manner, the frequency employed was that of the 10 H. P. alternator, that is 130 p. p. s.

3.

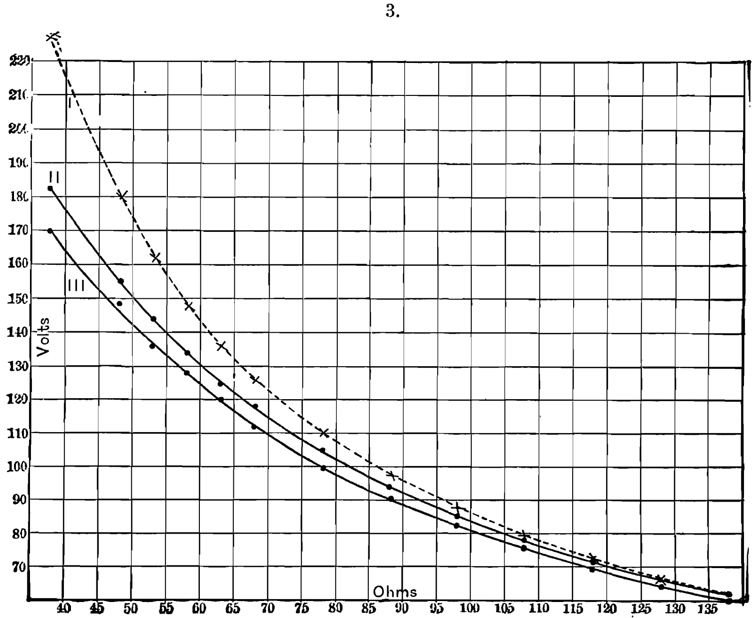


TABLE II.

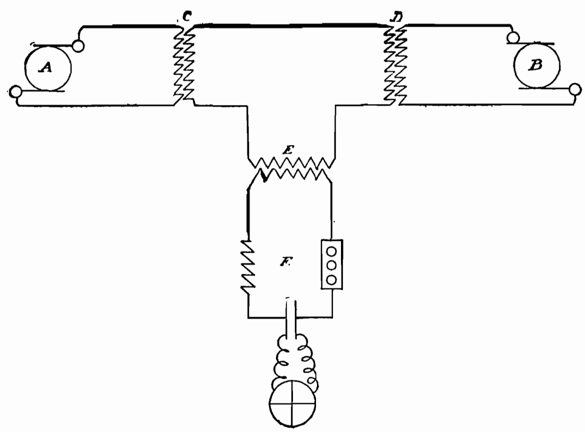
Resistance in ohms.	Voltmeter readings with a mica condenser.	Voltmeter readings with a paraffin condenser.	Theoretical value of voltmeter readings.
38	183	170	225·6
48	155	148	178·6
53	144	137	161·8
58	134	128	147·8
63	125	120	136
68	118	113	126
78	105	101	110
88	94	91	97·4
98	85	83	87·5
108	78	76	79·4
118	72·5	70	72·6
128	67	65	65·5
138	62	60	60

The experimental data from which these curves were plotted are given in Table II. Curve II was plotted from voltmeter readings obtained with a mica condenser, Curve III represents the corresponding readings obtained with a paraffin condenser and given in the third column of Table II. Curve I represents the theoretical curve, that is the curve which would have been obtained if the law of variation of the voltmeter readings with the resistance had been the same throughout as it was at low readings. On account of the damping effect due to dielectric viscosity in the condenser a deviation from the above mentioned hyperbolic relation was of course expected, but it was quite a pleasant surprise to find a perfect regularity of these deviations. These curves indicate a rapid increase in the dielectric damping with the voltage and also the superiority of mica to paraffin, especially at higher voltages. They also suggest that at low voltages and frequencies over a hundred periods per second this difference between the two substances becomes less and less marked. It was also found in a similar way that the damping effect of the magnetic viscosity of iron is small at low magnetizations, such, for instance, as would be produced by a telephonic current in a telephone receiver, and at frequencies which are well within the range of higher telephonic frequencies, say 750 periods per second.

Similar curves and similar results were obtained with higher harmonics. These experimental tests show, therefore, that the relative values of the amplitudes of the harmonics to that of the fundamental frequency are not seriously modified by the dielectric damping of the condensers, especially when one operates with moderate voltages as was the case in the following experiments.

b. Second test of the resonator indications.

4.



This test is represented graphically by diagram fig. 4. Two transformers C and D had their secondaries connected in series. The primary of the air-core transformer E formed part of their circuit. The secondary of the transformer E was a part of the resonator F. The transformer C, a Stanley 5 K.W. (closed magnetic circuit), was fed by the 10 H. P. alternator mentioned above (130 p. p. s.), the transformer D of induction coil type with a cylindrical core of fine iron wire was fed by a 1 H. P. alternator with slotted armature (278 p. p. s.) Both alternators were run simultaneously at full excitation. First, the primary circuit of the large alternator was broken, so that the current in the circuit CDE was due to the action of the small machine alone. The resonator detected a resonant rise of 240 volts at capacity ·407 M. F. and another of 150 volts at capacity ·044 M. F. These were evidently the fundamental and the first odd harmonic. Then the circuit of the small machine was broken and that of the large machine closed, so that the current in the resonator was due to the action of the large machine alone. The resonator detected a resonant rise of 220·1 volts at capacity 1·78 M. F. This corresponded to the fundamental frequency (130 p. p. s.) of the large machine. Finally both circuits were closed, so that the current in the resonator was due to the simultaneous action of the two machines. The same resonant rises of potential were detected by the resonator and at the same capacities as before, in perfect agreement with theory.

This experiment afforded another opportunity of testing the theory which underlies this resonance method of studying the wave curves of current and electromotive force. It is this: If two or more electromotive forces of different frequencies are impressed upon the resonator circuit and their resonant rises of potential are determined for a given resistance in this circuit, then according to theory the ratio of these rises should remain the same for all other resistances within the limits within which the periodicity of the circuit is practically independent of the ohmic resistance. Accordingly, the resistance of the resonator F, fig. 4, was varied gradually from 100 to 250 ohms (the self-induction of inertia coil in the resonator circuit was about ·75 Henrys) and the resonant rises of potential produced by the fundamental frequencies of the two machines (130 and 278 p. p. s.) were carefully determined for each particular resistance. The ratio of these rises remained constant to within five per cent but the deviations were now in one direction and now in the other. They were undoubtedly due to the variation in the excitation and the speed of the small machine, both of which depended on the potential of the electric mains of the College plant which, of course, could not be kept very constant for so long an interval of time as is necessary for this experiment, which was about 15 minutes.

These preliminary experimental tests demonstrate clearly that a resonator of the type given in fig. 1^b is quite capable of detecting all the frequencies that may exist in an alternating current wave, that its indications are in good agreement with the theory as far as the fundamental frequency is concerned and that it gives a fairly approximate idea of the relative strength of the harmonics.

(To be continued.)