ART. XXXVII.—*Influence of Temperature on the modulus of Elasticity of certain Metals;* by F. KOHLRAUSCH, Ph.D., and FRANCIS E. LOOMIS, Ph.D.

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THE results obtained by Wertheim,* in his investigations respecting the influence of temperature on elasticity, have thus far generally served as a basis for calculation. It will be observed in comparing these data, that for certain substances, especially for iron, the singular phenomenon manifests itself that for different temperatures the variations of the modulus undergo a change of sign. From 0° C. to 100° the modulus increases, and then from 100° to 200° decreases. This increase is certainly altogether contrary to what would naturally have been expected. Such a maximum, however, is still more remarkable. In view, therefore, of the uncertainty of the results of investigations thus far made known, and the importance of an accurate knowledge of the variations of the modulus of elasticity for different temperatures in many of the finer measurements, the present investigations were undertaken with a view if possible of obtaining more reliable results for some of the metals of greatest practical utility. For practical purposes it would suffice to determine the variations of elasticity within the limits of the more ordinarily occurring temperatures. This was partially accomplished by Kupffer, whose investigations we shall refer to hereafter. Nevertheless in consequence of the results obtained by Wertheim, it appeared of especial interest to determine again the variations in regard to their uniformity, as well as in regard to a change of sign. The observations were extended therefore to high temperatures, in general from 90° to 20° C., and in one case very nearly to 0°. Within these limits

* Pogg. Ann. Erg.. Band 2, S. 61; Ann. de Chimie, 3me S., T. 12, p. 443.

the variations in the elasticity can be determined from the observations with great accuracy. Since they show a remarkable uniformity, there can be no hesitation in assuming the resulting empirical formula as very nearly accurate, considerably beyond these limits.

It is scarcely necessary to observe that the investigations of Wertheim have afforded the most valuable material for a knowledge of the coefficient of elasticity. His method of observation, however, is little adapted to afford a solution of the question here under consideration. For Wertheim determined the absolute moduli of elasticity by means of the dilatation of bars and wires at different temperatures. The difficulties besetting an absolute determination of the variations of quantities amounting in all to only about $1\frac{1}{2}^{mm}$ are obvious. They were especially serious for the higher temperatures, since the imperfect heating apparatus of Wertheim could not be kept at a constant temperature for any considerable period of time, and, in consequence, the variations in length due to the fluctuations of temperature, inevitably exercised a serious influence on those arising from a change in elasticity. Wertheim himself designated these investigations as not rigorously accurate.* It would unquestionably have been more rational to confine the absolute determinations to ordinary temperatures, and to determine the influence of the temperature by means of other methods which here as in other cases, where it is a question merely of variations, are capable of a much greater sensibility. The influence of the fluctuations of the temperature on the length of the object will always occasion serious inconvenience in investigations on dilatation.

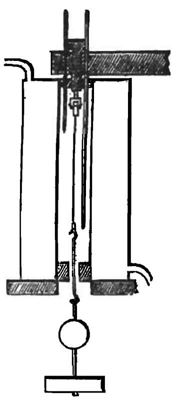
All these difficulties disappear, however, and at the same time the most accurate method of observation is obtained, by employing for investigation the torsion elasticity, whose choice is further to be recommended from the fact that torsion is so generally employed in measurements. If a wire is loaded with a weight, and set in vibration about its vertical axis, the reciprocal value of the square of the time of vibration affords a direct measure for the coefficient of the torsion of the wire. Since observations of the period of vibration are among the most accurate known in physics, the variations of elasticity may be thus determined with all the rigor desirable.

1. *Apparatus and mode of observation.*—The following dispositions were adopted for the practical application of this method. (See fig. in about $\frac{1}{12}$ natural size). The vibrating body consisted of a cylindrical leaden weight with vertical axis. Above it, attached to a brass rod, is a mirror for the purpose of observation with telescope and scale. The wire is clamped at each end

* Ann. de Chimie, 3me S., T. 12, p. 400 and 401.

between brass plates, of which the lower pair could be united to the weight by means of a upsilon rest, and the upper pair in like manner to a brass rod screwed into a wooden pivot. The latter was so secured in a solid support fixed to the wall as to be capable of rotation. The torsion vibrations were communicated to the weight by revolving the pivot by means of a lever. (Omitted in the figure).

2.



The space within which are enclosed the wire and a very considerable length of the connecting pieces is the hollow core of a double cylinder of tin, so constructed that the space inclosed between the two cylinders could be heated. For the determination of the temperature, three thermometers were prepared, such that the quicksilver reservoirs are at different distances from the zero point. When the graduated tubes project from the apparatus from the division 0°, the quicksilver reservoir of one of the thermometers is at a height corresponding to the middle of the wire and clòse beside it. The other two quicksilver reservoirs are at each end of the wire, and arranged symmetrically around it. The two latter thermometers alone are represented in the figure. The readings were made by means of a telescope.

The tin cylinder was tightly inclosed on all sides with a covering of felt of about a centimeter in thickness. It stood upon a support attached to the wall. The weight and mirror were contained in a box closed in front with a glass slide. These details are omitted in the figure.

It was originally designed to fill the vessel with water at different temperatures, but a simpler method, at first only employed to test what results might be expected, proved better adapted to the end in view. The felt envelop renders the loss of heat so gradual, that the empty apparatus, previously heated with steam, cools with sufficient slowness to permit the period of vibration at different temperatures to be observed with accuracy during the process. Accordingly the apparatus was heated for a series of observations by the admission of steam, until the temperature became tolerably stationary. The steam connection was then interrupted, and the temperature and vibrations alternately observed. Thus the variations of the elasticity of a wire for different temperatures are obtained in the course of a few hours. This simple mode of procedure affords at the same time the advantage of leaving the apparatus untouched from the beginning to the end of a series of observations, excepting only a slight turning of the wooden pivot, when the amplitude of the vibrations becomes too small. Several days were required for hot water to assume the temperature of the room, and it is not advisable to compare together observations made at long intervals, since the longer the interval the greater also becomes the danger that the conditions of the observation change in a manner to exercise on the period of vibration an influence feeble indeed, yet perhaps sensible, in view of the slight extent of the variations under consideration. For the same reason there are objections to employing water at different temperatures, since its renewal is scarcely possible without jarring the apparatus.

2. *Reduction of the observations.*—In the calculation of the data obtained from observations made with gradually diminishing temperatures, certain corrections are necessary, which render the reductions somewhat complicated. The manner in which the calculation was conducted, can be best explained by means of an example in detail. It is relative to a copper wire.

Each period of vibration was obtained according to the well known method of Gauss from two times of elongation, of which each depended in this case on six observations of the passage of the spider line through the position of rest. These times of elongation are contained in the first column of Table I. (See below). In the second column is contained the number of vibrations transpired between these times; in the third the period of vibration obtained by the division of the intervals in column one, by the numbers in column two.

TABLE 1.

Times of elongation.			Number of vibrations.	Period of vibration.	Temperature		
					Observed.	Corrections.	Corrected.
h	m	s		sec.	°	°	°
10	54	29·93					
			10	20·9970	79·4	−3·06	76·34
	57	59·90					
			9	20·9667	75·2	−2·95	72·25
11	1	8·60					
			9	20·9422	71·8	−2·83	68·97
	4	17·08					
			24	20·9000	66·4	−2·61	63·79
	12	38·68					
			29	20·8558	58·8	−2·28	56·52
	22	43·50					
			25	20·8248	52·7	−2·04	50·66
	31	24·12					
			39	20·7849	46·7	−1·78	44·92
	44	54·73					
			36	20·7614	41·3	−1·56	39·74
	57	22·14					
			58	20·7352	36·4	−1·36	35·04
12	17	24·78					
			92	20·7065	31·2	−1·15	30·05
	49	9·78					
			96	20·6861	27·0	−1·00	26·00
1	22	15·65					
			15	20·6767	25·2	−0·92	24·28
	27	25·80					

The temperatures, column 4, corresponding to the several periods of vibration, were obtained by a graphical representation of all the observed temperatures. The arithmetical mean was first taken of each three simultaneous readings of the thermometers, and then traced on coördinate paper, with the time of its observation as abscissa. The temperature for any moment can then be obtained from the curve, with an accuracy of about 0°·2 for the higher and about 0°·1 for the lower temperatures. The numbers in the 4th column are these mean temperatures for the several periods of vibration, i. e. they give the reading of the thermometers for the mean moment between the beginning and end elongation. When the interval of time is so considerable that the corresponding curve cannot be regarded as a straight line, the temperature τ would be thus obtained somewhat too low. To determine the true value, the arithmetical mean τ_0 was taken from the temperatures which correspond to the beginning and end time of elongation. The correct temperature is then $$\tau + \frac{\tau_0 - \tau}{3}.$$

Correction of the readings of the thermometers.—The numbers in the 4th column give the true temperature of the wire on condition, 1st, that the thermometers are accurate; 2d, that their temperature for each instant is the same as that of the space in which they hang. The heavier connecting pieces attached to each extremity of the wire extend so far within the heated space that the difference of temperature which might result in the wire from conduction may be neglected. It may be assumed, therefore, that the very fine wire follows sensibly the temperature of the space within which it is suspended. Since finally the readings of the upper and lower thermometers differed from the mean of all three in the highest temperatures only some 2°·5, the mean can be regarded as the temperature of the wire, without sensible error.

The above two conditions, however, must be fulfilled, or since they are not fulfilled, the readings must be corrected.

1. The three thermometers were therefore first calibred, and their fixed points determined. Thence tables were constructed containing the corrections with regard to a quicksilver thermometer theoretically accurate. The three tables united in one, permit then to apply the correction directly to the mean value of the three readings.

2. Secondly, in consequence of the considerable mass of quicksilver, the thermometers do not follow the temperature of the space so completely as can be assumed of the fine wire. Now according to well known laws, the amount of heat lost by a body, and consequently therefore also the variation of its temperature, is proportional to the difference of its temperature and that of the surrounding space. Since in the present case, the velocity of the variation of temperature of the thermometers is known for each moment, it is only necessary to determine once for all the relation of the true temperature to that indicated by the thermometers, to be able to calculate the difference, i. e. the correction of the readings. For this determination a very delicate thermometer was employed, whose cylindrical quicksilver reservoir possessed a diameter of only some 3^{mm}, and contained in all less than 1·5 gr. of quicksilver. It

was assumed that this thermometer follows the gradually diminishing temperature of the air to within a negligable fraction, and the three thermometers were compared with it, under conditions similar to those existing in the observations on elasticity. All three thermometers were disposed at an equal height with the thermometer of comparison, in the gradually diminishing temperature of the hollow space of the tin cylinder, and simultaneous readings were made. They were also compared together in the ordinary manner in water of different temperatures. From these observations, which were conducted by Mr. Grotrian, it resulted that the mean of the readings of the three thermometers remained $2^\circ{\cdot}02$ behind the temperature of the thermometer of comparison, when the velocity of the diminishing temperature amounted to 1° in 1 min. The correction of the readings due to the mass of quicksilver is therefore

$$2{\cdot}02\,\frac{d\tau}{dt},$$

where τ denotes the temperature of the thermometers, and t the time expressed in minutes. We obtain thus the temperatures which would have been observed, if thermometers had been employed containing the same amount of quicksilver as the thermometer here employed for comparison.

3. Finally a *third correction* is necessitated by the fact that the column of quicksilver of the thermometers, beginning with the zero point of graduation, possessed a lower temperature in consequence of projecting outside of the apparatus. There exists as yet no simple method of determining the mean temperature of the projecting column. It was assumed to have possessed the temperature of the air to the extent of three parts, and that of the interior of the apparatus one part. If τ_0 denotes the temperature of the room, the correction of the reading τ is therefore $-\frac{3}{4}\tau(\tau-\tau_0)\;0{,}00016.$

This value cannot be far removed from the truth. It coincides with the difference which the determination of the boiling point afforded, according as the thermometers were placed in the steam only to the zero point of graduation, or to the boiling point. The uncertainty of this assumption is at all events of small importance, since the whole correction for the highest temperature (90°) amounts only to $0^\circ{\cdot}8$.

Thus, three corrections are to be applied to the readings of the thermometers. The first, which relates to the ordinary errors, is a function of the temperature; the second, in consequence of the mass of the quicksilver, is a function of the velocity of the diminution of the temperature; the third, finally, in consequence of the column of quicksilver projecting from the apparatus, depends on the difference of the outer and inner temperatures. Nearly the same conditions prevailed, however, during all the series of observations. The temperature of the room differed from the mean at most 5°, and consequently the velocity of the diminution of the temperature of the apparatus depends, within negligable differences, on the inner temperature alone, as is confirmed indeed by the observations themselves. Thus the true velocity of the diminution for different temperatures could be determined from a few series of observations, whence by multiplication with 2·02, (see p. 355) the correction itself is obtained as function of the temperature.

In like manner, for the calculation of the third correction in the formula (p. 355), it was permissible to take $\tau_0 = 20°$ as a mean for all the observations.

The corrections may now be very much simplified by uniting all three in tabular form, viz: for reading,

0°	10°	20°	30°	40°	50°	60°	70°	80°	90°

the correction is

0°·0*	–0°·4	–0°·7	–1°·1	–1°·5	–1°·9	–2°·3	–2°·8	–3°·1	–3°·2

The interpolation for intermediate temperatures was accomplished by means of a graphical representation of this table. In one case, however, when the apparatus was filled with water, and the velocity of refrigeration could be neglected, the corrections 1 and 3 were applied separately.

Correction of the observed periods of vibration.—1. In the present investigations we are seeking the variations of the modulus of elasticity caused by a change of temperature, whereas we observe the directive force of the whole wire, which, in consequence of the heating, experiences a dilatation,as well as a change in its elasticity. Since this directive force of the wire depends on the length and section as well as on the modulus of elasticity, it is necessary to investigate what corrections are thereby necessitated.

It is an interesting fact that no correction at all is necessary. The dilatation in length and breadth completely neutralize each other.

If we denote by

l the length of the wire,
r the radius,
m the mass of the unity of length,
K the moment of inertia of the vibrating weight,
t the time of vibration,

* It is merely an accidental coincidence, that there is no correction to be applied to the reading 0°. The mean of the zero points of the three thermometers happens to be perfectly accurate.

D the directive force of the wire, i. e. the moment of revolution which it exercises for a torsion angle of $\frac{180^\circ}{\pi} = 57^\circ{\cdot}296$.

ε the modulus of torsion of the substance constituting the wire,

we have $$D = \frac{\pi^2}{t^2} K$$

and $$\varepsilon = \frac{Dl}{gmr^2} = \frac{\pi^2 K}{g} \frac{l}{mr^2 t^2}, \tag{1}$$

where g denotes the acceleration due to gravity, and ε is the number, the quintuple of which (according to the theory of Poisson, i. e. when the ratio of the transversal contraction to the longitudinal dilatation is $=\frac{1}{4}$) multiplied by g gives the square of the velocity of sound in the substance.

When now in consequence of a change of temperature, the elasticity and dimensions of the wire vary, the time of vibration t changes also. If we denote the variations of ε, l, m, r and t, occurring for a small change of temperature, by $d\,\varepsilon$, $d\,l$, $d\,m$, $d\,r$ and $d\,t$, we obtain by differentiating (1) logarithmically,

$$\frac{d\varepsilon}{\varepsilon} = \frac{dl}{l} - \frac{dm}{m} - \frac{2dr}{r} - \frac{2dt}{t}.$$

But on the supposition that the wire experiences an equal dilatation in all directions on account of the temperature $\frac{dl}{l} = \frac{dr}{r}$. Also since m denotes the mass of the unity of length, obviously

$$-\frac{dm}{m} = \frac{dl}{l} = \frac{dr}{r}.*$$

Therefore $$\frac{d\varepsilon}{\varepsilon} = -2\frac{dt}{t}. \tag{2}$$

The modulus of torsion ε is therefore without correction reciprocally proportional to the square of the time of vibration.

The definition of the modulus of elasticity assumed here as a basis, denotes, with regard to the dilatation, the weight which must be suspended to a wire, whose unity of length is the same as the unity of its mass, to double the length. In practice it is customary, though frequently less convenient, to employ the section instead of the weight of the unity of length. In such case, the above expression (1) must be multiplied further by the density of the substance, and the coefficient of the cubical dilatation 3α must be subtracted from formula (2), so that

$$\frac{d\varepsilon}{\varepsilon} = -\frac{2dt}{t} - 3\alpha.$$

* $m \,.\, l = (m + dm)(l + dl)$

$0 = l\,dm + m\,dl$

$\frac{dm}{m} = -\frac{dl}{l}.$

2. A small correction is necessary when the moment of inertia of the vibrating weight changes in consequence of fluctuations in the temperature. In general it is inconsiderable in the same series of observations. The temperature in the box containing the vibrating weight was observed from time to time, and the period of vibration corrected when changes of temperature occurred. This was accomplished by multiplying the observed time of vibration by $1-0,00003.\Delta\tau$, where 0,00003 denotes the coefficient of the linear dilatation of lead, and $\Delta\tau$ the variation of temperature.

3. Theoretically the time of torsional vibrations should be independent of the amplitude. In fact in the present observations on the brass wire, no influence was perceptible. The iron wire, hereafter to be discussed, was especially investigated with respect to such an influence, however, and showed a manifest, though very small, increase of the time of vibration for an increment of amplitude. The increase appeared to be proportional to the latter, and on this supposition amounted to $0{\cdot}000037^{\text{sec.}}$ or $\frac{1}{500000}$ of the whole time for one division of the scale ($\frac{1}{7000}$ to one degree of arc). A series of observations on the copper wire were incidentally of a nature to afford the means of calculating the increase approximately. It amounted to $0{\cdot}000015^{\text{sec.}}$ for one division of the scale. The times of vibration were reduced by means of these numbers to an infinitely small amplitude.

Although it would be of interest to examine this phenomenon more minutely, it hardly enters within the scope of the object in view in these observations. For in consequence of the inconsiderable amplitudes which we employed (at most 200 divisions of the scale, or some 3°) the correction in extreme cases amounted only to $0{\cdot}005^{\text{sec.}}$, and it is almost completely compensated in its influence on the final result.

3. *Calculation of the coefficient of temperature.*—We have thus far obtained a series of temperatures with the corresponding times of vibration. From each two pairs of corresponding values may be readily derived the variation of the elasticity for 1°. We express it as a function of the total elasticity at 0°. If ε_0 designates the modulus of elasticity at 0°, and ε_1, ε_2, the moduli corresponding to the temperatures τ_1, τ_2, the desired decrease for 1° is found from formula (1) to be

$$\frac{1}{\varepsilon_0}\,\frac{\varepsilon_1-\varepsilon_2}{\tau_2-\tau_1} = t_0^2\,\frac{\dfrac{1}{t_1^2}-\dfrac{1}{t_2^2}}{\tau_2-\tau_1},$$

where t_1 and t_2 designate the times of vibration for the temperatures τ_1 and τ_2, and their squares are inversely proportional to the modulus of elasticity.

The following Table 2 contains in the second column the preceding diminutions of the modulus of elasticity for 1° taken from Table 1, by combining in it each two adjoining values. The time of vibration t_0 for 0° was obtained from the graphical representation of the observations by prolongation of the curve, a method which affords sufficient accuracy. Thus it was found that $t_0 = 20{\cdot}58$ sec. In the first column is contained the mean of the two temperatures at which the times of vibration were observed.

TABLE 2.

Temperature.	Decrease of the coefficient of elasticity for 1° as func. of the coeff. of elast. at 0°. Observed.	Calculated.	Observ.—Calc.
74°	0,00068	0,00068	±0,00000
71	69	67	+ 2
66	75	65	+ 10
60	56	62	− 6
54	50	60	− 10
48	65	57	+ 8
42	43	55	− 12
37	53	53	± 0
33	55	51	+ 4
28	48	49	− 1
25	0,00052	0,00047	+0,00005

The numbers of the second column can be regarded as the values of $-\frac{1}{\varepsilon_0}\frac{d\varepsilon}{d\tau}$ for the temperature τ of the first column. It will be observed that this mode of observation affords the decrease of the modulus of elasticity for 1° with tolerable accuracy, even for the times of vibration for two temperatures differing only by a few degrees. (The greatest difference is 7°, the smallest 2°). This decrease for the brass wire is noticeably greater for the higher temperatures than for the lower.

The numerical law of the variation is to be derived from all the observations. We seek to represent it by the formula

$$\varepsilon = \varepsilon_0 \, (1 - a\,\tau - b\,\tau^2).$$

It would obviously be more in accordance with the laws of the calculus of errors to determine the values of the coefficients ε_0, a and b by means of least squares from all the observed values of ε or $\frac{1}{t^2}$ for all temperatures. This would prove, however, when applied to all the series of observations, excessively laborious. It would be hazardous to unite previously together all the observations made on the same wire, since the time of vibration for the same temperature appears to be subject to slight variations from one day to another, probably in consequence of the considerable heating to which the wire is subjected.

By differentiating the above equation we obtain

$$-\frac{1}{\varepsilon_0}\frac{d\varepsilon}{d\tau} = a + 2b\tau.$$

The left hand expression denotes the values contained in the second column of Table 2. If therefore we substitute these values in the above equation, the coefficients a and b can be readily calculated.

By means of least squares there results for the brass wire

$$a = 0{,}000368 \qquad 2b = 0{,}00000425$$

by means of which the figures in the third column were calculated. In view of the small differences of temperature the accordance must be regarded as satisfactory. It appears still more manifest when the time of vibration itself is calculated. Since a and b are known, the most probable value of the time of vibration for 0° may also be readily determined by means of least squares. The calculation gives $t_0 = 20{\cdot}5696^{\text{sec.}}$

The times of vibration calculated from this value of t_0, together with the times observed, are given in the following Table 3. The difference between the observed and calculated values of the times of vibration amounts at most to $\frac{1}{120}$ of a second, and indicates sufficiently the admissibility of the simplified method of reduction, although the regularity in the sign of the differences would seem to indicate that the rigorous method would afford a still greater accordance.

TABLE 3.

Times of vibration.

Temperature.	Observed.	Calculated.	Observ.—Calcul.
76°·34	$20^{s}{\cdot}9970$	$20^{s}{\cdot}9986$	$-0^{s}{\cdot}0016$
72·25	20·9667	20·9680	−0·0013
68·97	20·9422	20·9443	−0·0021
63·79	20·9000	20·9078	−0·0078
56·52	20·8559	20·8592	−0·0033
50·66	20·8248	20·8218	+0·0030
44·92	20·7849	20·7869	−0·0020
39·74	20·7614	20·7569	+0·0045
35·04	20·7352	20·7309	+0·0043
30·05	20·7065	20·7041	+0·0024
26·00	20·6861	20·6837	+0·0024
24·28	20·6767	20·6751	+0·0016

4. *Temperature and modulus of elasticity of iron, copper and brass.*—The method of observation and reduction, already described and applied to an example, was employed for wires of iron, of copper and of brass. All three wires were hard drawn and of a diameter between 0·2 and $0{\cdot}3^{\text{mm}}$. The copper wire was composed of copper precipitated electrolytically. The iron

and brass wires are the ordinary wires of commerce wound on spools.

To abbreviate the calculation, the times of vibration and the temperatures of each series of observations are united first in groups by taking the mean, and especial care was taken to allow about the same weight to all the several coefficients to be derived. The observed data corrected according to the rules of chapter 2, are contained in the following Table 4, in which the Roman characters denote the chronological succession of the several series of observations.

TABLE 4.

Iron.			Brass.			Copper.		
	Temp.	Time of Vibration.		Temp.	Time of Vibration.		Temp.	Time of Vibration.
	°	s.		°	s.		°	s.
I	76·34	17·6154	I	69·01	20.9412	I	74·06	12·4444
	56·87	17·5282		47·95	20.8067		47·96	12·3476
				28·85	20.7012		55·20	12·3696
II	76·55	17·5950					40 80	12·3221
	55·75	17·5087	II	72·14	20.9105		24·04	12·2667
	42·42	17·4553		48·78	20.7600			
	32·11	17·4150				II	62·60	12·3852
	21·91	17·3679	III	70·28	20·8694		22·05	12·2465
				49·11	20·7368			
III	78·73	17·5920		36·24	20·6669	III	77·27	12.4303
	59·72	17·5130		27·23	20·6193		56·16	12·3546
	20·97	17·3583					40·79	12·3028
			IV	50·05	20·7333		23·37	12·2413
IV	49·57	17·4715		42·81	20·6915			
	37·80	17·4279		34·59	20·6425			
	21·95	17·3627						
			V	19·43	20·5428			
				5·05	20·4737			

The following Table 5, in which the numbers are arranged according to the temperatures, is obtained by reducing the values contained in Table 4 in the same manner as those of Table 2 of the example.

The coefficients a and b are derived by means of least squares from the values contained in the third and fourth columns of Table 5, as in the example. Designating by ε_0 the modulus of elasticity at 0°, the modulus for the temperature τ is found to be,

for iron, $\varepsilon = \varepsilon_0(1 - 0{\cdot}000447\tau - 0{\cdot}00000012\tau^2)$
for copper, $\varepsilon = \varepsilon_0(1 - 0{\cdot}000520\tau - 0{\cdot}00000028\tau^2)$
for brass, $\varepsilon = \varepsilon_0(1 - 0{\cdot}000428\tau - 0{\cdot}00000136\tau^2)$

The numbers of the next to the last column of Table 5 were calculated with these values. The probable error of each value of the decrease of the coefficient of elasticity for 1° for a given temperature calculated from two groups of observations amounts accordingly to $\pm$ 0·000014.

TABLE 5.

		Temp.	Decrease of the coeff. of elastic. for an incr. of temp. of 1° as func. of the coeff of elast. at 0°. Observed.	Calcul.	Observ. — Cal.
Iron.	IV	29°·9	0·000467	0·000454	+ 0·000013
	II	32·2	482	455	+ 27
	III	40·3	450	457	— 07
	IV	43·7	416	458	— 42
	II	43·9	445	458	— 13
	II	66·1	459	463	— 04
	I	66·6	494	463	+ 31
	III	66·9	459	464	— 05
Copper.	III	32·1	0·000566	0·000538	+ 0·000028
	I	32·4	528	538	— 10
	II	42·3	542	544	— 02
	I	48·0	521	547	— 26
	III	48·5	531	547	— 16
	I	61·0	578	554	+ 24
	III	66·7	556	557	— 01
Brass.	V	12·2	0·000466	0·000461	+ 0·000005
	III	31·7	504	514	— 10
	I	38·4	523	532	— 09
	IV	38·7	570	533	+ 37
	III	42·7	514	544	— 30
	IV	46·4	556	554	+ 02
	I	58·5	593	587	+ 06
	III	59·7	584	590	— 06
	II	60·5	599	592	+ 07

If the other definition of the modulus of elasticity is employed (as was done by Wertheim) referring to the section, and not, as above, to the mass of the unity of the length of the wire, the coefficients of the temperature will be slightly altered in accordance with the observation on page 357. If the coefficients of dilatation for 1° are assumed to be for iron $=0·000012$; for copper $=0·0000175$; and for brass $=0·000019$, the coefficients of τ become,

for iron,	0·000483,
for copper,	0·000572,
for brass,	0·000485.

The coefficients of the quadratic terms remain sensibly unchanged.

It is evident from the results of these observations that the mean variation of the elasticity for the three metals investigated differs but little from that of the temperature. For a change of temperature from 0° to 100° the diminution is for

Iron,	4·6	and	5·0	per cent.
Copper,	5·5	"	6·0	"
Brass,	5·6	"	6·2	"

where the numbers of the second column relate to the second definition of the modulus of elasticity.

If the variation in elasticity is compared with the influence exercised by the temperature on other properties of bodies it will be remarked that it is much greater than the cubical dilatation as well as the variations of refraction. It is of about the same order as the variation of the permanent magnetism caused by the temperature, as well as that of specific heat. The increase of the galvanic resistance on the other hand is much greater.

It results further from the sign of the quadratic term that the variation of elasticity for all three metals is the most rapid in the higher temperatures. While, however, the increase for iron is almost imperceptible, and is also very small for copper, it is quite considerable for brass. The decrease for 1° is for

	At 0°.	At 50°.	At 100°.	
Iron,	0·0447	0·0459	0·0472	per cent.
Copper,	0·0520	0·0548	0·0576	"
Brass,	0·0428	0·0564	0·0699	"

It will be observed that the differences in the variation of the coefficient of elasticity for the three metals investigated, are in the order of the height of their melting points.

The results of these investigations show no trace at all of the remarkable phenomenon of a maximum, alluded to at the beginning of this article, which would seem to be indicated for iron by the investigations of Wertheim. If, therefore, different varieties of iron do not manifest a totally different behavior, or if the modulus of the longitudinal elasticity does not undergo a very different change in consequence of temperature from that of the torsional elasticity, this anomaly must be accounted for by the imperfect method of observation employed by Wertheim. This supposition is confirmed further by the observations of Kupffer (see below), as well as by a very simple experiment. If, namely, two tuning forks are in vibration, and one of them is heated, the number of vibrations changes in the manner demanded by the assumption of a decrease of elasticity for increasing temperatures.

It is to be remarked here that Wertheim's calculation of the heat generated by condensation during vibration, loses thus all its value.*

Observations of Kupffer.†—These investigations appear to be much less generally known than they deserve, for they contain much varied and valuable material for practical use. Kupffer's observations are in general on bars vibrating trans-

* Pogg. Ann., Bd. 11, S. 32.

† Mem. de l'Acad. de St. Petersb. 1856, 6 ser. T. vi, p. 400.

versally at the temperatures of about $-10°$, $+15°$ and $+80°$. Unfortunately, however, very long intervals (at least a year) occur between the observations, and it would seem that he did not always employ the same bars, so that his investigations permit but a limited conclusion in regard to the law of the variations of elasticity as modified by temperature. Finally, the dilatation caused by temperature was disregarded in the numerical values given by Kupffer, a source of serious errors in view of the method of observation which was employed.

A few of the results derived by Kupffer were obtained by means of torsional vibrations. Among these are the coefficients of temperature for the elasticity of an iron wire (piano cord), of a copper wire and of a brass wire. After having applied to these values the correction required for the dilatation of the vibrating disc, the decrease of the elasticity for 1° expressed in parts of the total elasticity is found to be

for	iron,	0·00055	(" mem." S. 446)
"	copper,	0·00082	S. 464
"	brass,	0·00039	(S. 467)

These values agree tolerably well with those found by us, with the exception of the copper wire, in regard to which it is possible that the divergence may be caused by the circumstance that we employed a chemically pure metal.

5. *The absolute moduli of elasticity of the iron, copper and brass wires calculated from torsional vibrations and from the velocity of sound.*

To determine from our investigations the absolute moduli of torsional elasticity, it is necessary to know the dimensions and mass of the wires, as well as the moment of inertia of the vibrating weight.

The latter is to be calculated from the mass of the perforated lead cylinder $=1818$ grs. and the interior radius $=0·34$ c. m. and the exterior radius $=5·07$ c. m. It is found to be

$$\frac{1818}{2}\left(\overline{5·07}^2+\overline{0·34}^2\right)=23470 \text{ grs. } \square \text{ c. m.}$$

To this must be added the moment of inertia $=40$ grs. □ c. m. of the connecting pieces and the mirror, which was calculated from their size and form. The total moment of inertia therefore is $\qquad K=23510$ grs. □ c. m.

Further was determined

	Iron.	Copper.	Brass.
The length of the wire,	$l=20·80$	20·75	20·80 c. m.
The mass of one c. m. length,	$m=0·00301$	0·00655	0·00403 grs.
The density,	$\Delta=7·82$	9·00	8·41
The time of vibration at 20°,	$t=17·35$	12·23	20·55 sec.
The radius of the wire,	$r=0·0111$	0·0152	0·0123 c. m.

The modulus of torsion T is derived from these data by means of the formula, p. 357,

$$T=\frac{\pi^2 K}{g}\frac{l}{mr^2t^2}.$$

It is found to be

for iron	=444 kilometers.	
" copper	=217	"
" brass	=190	"

If it is desired to follow the definition customary in practice, and to refer the modulus to the section instead of to the mass of the unity of length, these numbers must be multiplied by the corresponding densities; whereby they become

for iron,	3470 $\frac{\text{kilogrammes}}{\text{quadratmillimeter.}}$	
" copper,	1950	"
" brass,	1600	"

The velocity of sound was measured in the same species of wires by stretching them in a Weber's monochord, and by varying the length, comparing the longitudinal tone with tuning forks. The normal tone of the set of tuning forks was determined by comparison with two normal tuning forks of Appunn.

The modulus of dilatation of the wire (i. e. the length of a wire of the same species possessing the weight necessary to double the original length), is obtained from the velocity of sound c by means of the formula $A=\frac{c^2}{g}$, where g denotes the acceleration due to gravity. The value A′ ordinarily employed in practice, and made use of by Wertheim, (i. e., the weight which hung on a wire whose section is unity would double the length), is obtained as above by multiplication with the density. Thus it was found

	Velocity of sound.	Modulus of elasticity. A.	A′.
Iron,	5050 meters.	2580 kilometers.	20310 $\frac{\text{kgr.}}{\text{quad. mill.}}$
Copper,	3640 "	1350 "	12140 "
Brass,	3380 "	1170 "	9810 "

Wertheim gives A′ for iron wire =19445, copper =12536, brass =9000 $\frac{\text{kgrs.}}{\square\ \text{m.m.}}$

According to these observations, the moduli of torsion bear to the moduli of dilatation, for all three wires very nearly the same ratio. The ratio $\frac{A}{T}$ is

for iron,	5·85
" copper,	6·23
" brass,	6·15
Mean,	6·07

The deviations of these values from the mean are perhaps due to errors of observation, especially in the determination of the densities, since in order to be sure of investigating the same material as in the original experiments, but small masses could be employed. It is possible too that the deviation of the mean from the whole number 6, is to be sought in these errors.

Without entering on a more extended investigation of this point, and without discussion as to whether the customary theory of elasticity is applicable to such thin wires, or whether the lack of isotropy, or the relations of superficies, forbid this application, it may be remarked that according to the theory

$$\frac{A}{T} = 4(1+\mu),$$

where μ denotes the ratio of the transversal contraction to the longitudinal dilatation. By a comparison of the torsional and longitudinal vibrations, there would result accordingly for our three wires very nearly the same value for μ. If it is assumed that $4(1+\mu)=6$ there results $\mu=\frac{1}{2}$. This is the extreme value permitted by the theory, which would correspond to a volume unchanged by dilatation.

It may suffice to have indicated the fact. It would lead us too far from the chief object of our investigation to develope the subject more in detail.