PHYSICAL AXES OF REFERENCE AND GEOMETRICAL AXES OF REFERENCE FOR QUARTZ.

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ABSTRACT. Coördinate axes OX, OY, and OZ used in equations involving piezo-electric, elastic, and other physical properties of crystals may be called *physical axes of reference.* These are orthogonal axes for hexagonal crystals as well as for crystals of the other crystal systems.

Coördinate axes of reference for designating crystal faces may be distinguished as *geometrical axes of reference.* For crystals of the hexagonal system the following sets of geometrical axes of reference (axial systems) have been employed:

$a_1 : a_2 : a_3 : c$; $a_1 \wedge a_2 = a_2 \wedge a_3 = 60°, a_1 \wedge c = 90°$ (Weiss)

$a_1 : a_2 : a_3 = 1$; $a_1 \wedge a_2 = a_2 \wedge a_3 = a_1 \wedge a_3 = \alpha$ (Miller)

$a_1 : a_2 : a_3 : c$; $a_1 \wedge a_2 = a_2 \wedge a_3 = 120°, a_1 \wedge c = 90°$ (Bravais)

$a : b : c = \sqrt{3 : 1} : c$; $a \wedge b = b \wedge c = a \wedge c = 90°$ (Schrauf)

Only in the case of the orthohexagonal axes of reference of Schrauf do the physical and geometrical axes of reference coincide but the orthohexagonal axes are not suitable for geometric axes of reference.

The Bravais axes of reference are to be preferred to the Miller axes since the unit cell of the space lattice of quartz is a hexagonal prism-pinakoid combination rather than a rhombohedron.

Although a left-handed axial system may be advantageous for physical axes of reference in dealing with left quartz, the same set of geometrical axes of reference should be used for both left and right quartz.

In drawing enantiomorphous pairs of crystals (either plans, elevations, or clinographic drawings) left-handed crystals may be rotated to the right (clockwise) instead of to the left (counter-clockwise) as ordinarily.

INTRODUCTION.

IN equations involving the piezo-electric and elastic properties of quartz an orthogonal set of coördinate axes of reference is used (1) (2). The optic axis is selected as the Z-axis; one of the three polar electric axes, which are axes of 2-fold symmetry (A_2), is taken as the X-axis, and a line mutually normal to the optic axis and to the particular electric axis chosen is made the Y-axis. The latter-mentioned direction has in recent years been called the mechanical axis. The accompanying drawing (Fig. 1) shows in plan and side elevation a typical quartz crystal (forms $m\{10\bar{1}0\}$, $r\{10\bar{1}1$, and $z\{01\bar{1}1\}$) with an oriented plate usually known as the X-cut since it is normal to an X-axis. It will be recalled that alternating voltage applied along the electric or X-axis of such a plate sets up a vibration (expansion and contraction) parallel to the mechanical or Y-axis and conversely, stress applied in the direction of the mechanical axis produces an electric charge on the sides of the plate in the direction of the electric axis. The frequency of the electric charge is largely a function of the dimensions of the plate(3).

A set of coördinate axes such as X, Y, and Z[1] may be called *physical axes of reference* to distinguish them from coördinate

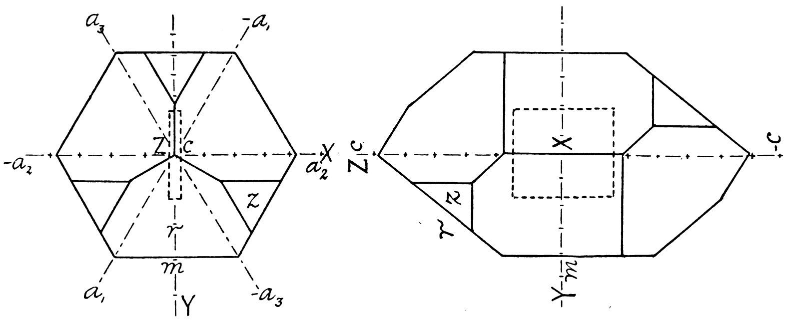


Fig. 1. Plan and side elevation of a typical quartz crystal with X-cut plate showing physical axes of reference X, Y, and Z and geometrical axes of reference, a_1, a_2, a_3, and c.

axes used in denoting the position of crystal faces which may be called *geometrical axes of reference.*[2]

In Fig. 1, the geometrical axes of reference of Bravais, a_1, a_2, a_3, and c are represented by dot-and-dash lines. The physical axes of reference X, Y, and Z are indicated by dash-with-vertical-dot lines. Where geometrical and physical axes of reference coincide as in the case of the c-axis and the X-axis a dash-with-cross-dot line is used. Orthogonal axes are always employed for physical axes of reference in all crystal systems.[3]

Crystal morphology is unusual in that a considerable variety of axes of reference are required. In the customary treatment

[1] If capital letters are used for these axes there need be no confusion with the lower-case letters x and z used in designating crystal faces and forms.

[2] The term "crystallographic axes" often used for axes of reference should be avoided since there are various other axes involved in geometrical crystallography such as axes of symmetry and zone-axes which are no less important.

[3] These are usually given as OX, OY, and OZ as in analytic geometry, but Wooster(2) prefers OX_1, OX_2, and OX_3.

of geometrical crystallography these are almost taken for granted and there is not enough discussion of their variation. The chief emphasis placed upon axes of reference has been their use in defining crystal systems but entirely too much stress has been put on this relation. It seems better to define crystal systems in terms of zones (4, p. 38).

It is my purpose here to inquire into different sets of geometrical axes of reference, or axial systems[4] as Story-Maskelyne (5, p. 27) called them, used at various times for crystals of the hexagonal system which may be most simply defined as including all crystals with a single hexagonal zone (4, p. 39).

THE AXIAL SYSTEM OF WEISS.

Weiss(6), who shares with Mohs the honor of establishing the crystal systems in the early part of the nineteenth century, made use of four axes of reference, three in one plane (a_1, a_2, a_3) at angles of 60° to each other and a fourth (c) normal to the plane of the other three (see Fig. 2). At that period this

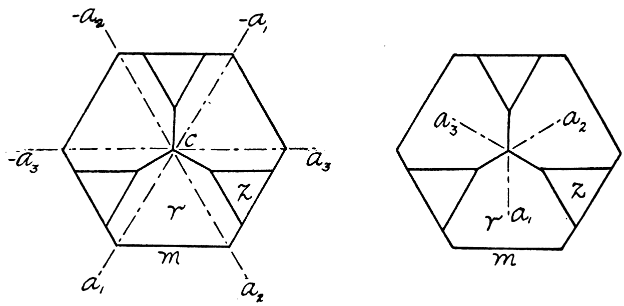


Fig. 2. Plan of a quartz crystal with the forms $r\,\{10\bar{1}1\}$, $z\,\{01\bar{1}1\}$, and $m\,\{10\bar{1}0\}$, showing the Weiss axes of reference.

Fig. 3. Plan of a quartz crystal showing the Miller axes of reference.

must have been a bold suggestion but time has proved the wisdom of his proposal. He selected axes to suit the crystal rather than the reverse.

THE AXIAL SYSTEM OF MILLER.

Miller(7) adopted the method of indicating crystal faces by the index-symbols which bear his name, following the suggestion of Whewell(8), his predecessor at Cambridge, who is chiefly

[4] Axial systems, it should be remarked, have no necessary connection with crystal systems.

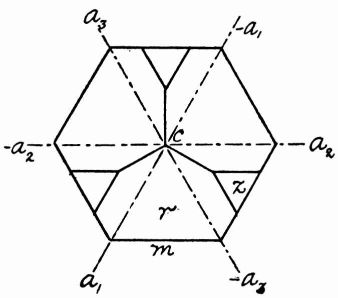
noted for his *History of the Inductive Sciences.* Instead of using four axes of reference for hexagonal crystals, Miller chose three axes parallel to the polar edges of a rhombohedron (see Fig. 3). Since these axes of reference are interchangeable they are designated as a_1, a_2, and a_3 and the interaxial angle α becomes the geometrical constant instead of the ratio $a:c$. Miller placed all hexagonal crystals in his rhombohedral system and did not recognize the hexagonal system even in his later work(9). For the general form of the dihexagonal dipyramidal class he used the double symbol $\{hkl\},\{efg\}$. In spite of the fact that the indices h,k,l, and e,f,g are mathematically related, this feature is so objectionable that it has been abandoned for crystals of the hexagonal subsystem by practically all modern crystallographers.

THE AXIAL SYSTEM OF BRAVAIS.

Bravais(10), who may be considered the founder of structural crystallography, remedied the difficulty of the double symbol necessary for some hexagonal crystals by using the four axes of reference of Weiss but took them in a different order. He placed the positive ends of the lateral axes a_1, a_2, and a_3 at angles of 120° (instead of 60°) as shown in Fig. 4. This furnishes the four-index symbol $\{hk\bar{i}l\}$ which is now known as the Bravais symbol. By selecting lateral axes at 120° instead of at 60°, crystals belonging to the rhombohedral subsystem (4, p. 75) which greatly predominate over those of the hexagonal subsystem (4, p. 75) are adequately treated. Bravais symbols are now much more extensively used than Miller symbols for crystals of the rhombohedral subsystem. Ungemach, Alsatian crystallographer, went so far as to express the opinion that the Miller three-index symbols should not be used for any crystals of the hexagonal system. This is an extreme view; it would seem preferable to employ Miller symbols in cases where the unit-cell of the space-lattice is a rhombohedron (Γ_{rh}) and Bravais symbols in cases where the unit-cell is a hexagonal prism-pinakoid (Γ_h) combination. This would mean the employment of both Bravais symbols and Miller symbols in the list of forms for the five classes of the rhombohedral subsystem as recommended by the writer (4, pp. 85-86) and now adopted in the new (seventh) edition of Dana's *System of Mineralogy.* The use of Miller indices brings out the relation of rhombohedral crystals to isometric crystals; this is often lost sight of (4, p. 85). Miller indices may also be important in that they demonstrate that the faces of any crystal whatsoever may be represented by a three-index symbol.

THE AXIAL SYSTEM OF SCHRAUF.

Schrauf(12), one of the school of crystallographic physicists of Vienna, introduced orthogonal axes of reference for crystals of the hexagonal system. This he did by selecting one of the



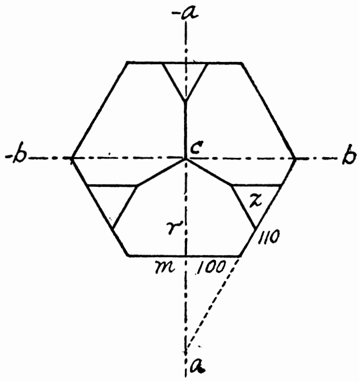


Fig. 4. Plan of a quartz crystal showing the Bravais axes of reference.
Fig. 5. Plan of a quartz crystal showing the Schrauf axes of reference.

three axes of 2-fold symmetry as the b-axis and a line normal to this axis and also normal to the vertical axis as the a-axis. The unit on the b-axis is 1 and the unit on the a-axis is $\sqrt{3}$ or 1.732. The c-axis is normal to both the a-axis and the b-axis. The axes of reference are shown on a plan view of a quartz crystal (Fig. 5). Schrauf's change of the name of the hexagonal system to orthohexagonal system is an interesting demonstration of the undue emphasis placed upon geometrical axes of reference in defining crystal systems.

In the Schrauf method the geometrical axes of reference coincide with the physical axes of reference and at first glance this may appear to be an advantage, but an insuperable objection is that a form such as the hexagonal prism (m) requires a double symbol {100}{110} as may be seen from Fig. 5. It is now generally agreed among crystallographers that any form must be represented by a single form-symbol and furthermore that all the face indices of a form must be permutations of one of the faces of the form. This, it will be recalled, was one of the reasons for rejecting Miller's scheme of using three-index symbols for crystals of the hexagonal subsystem. Since the vertical axis in the hexagonal system is always either a three-fold or a six-fold axis of symmetry it is important that the lateral axes of reference should be symmetrically placed with respect to crystal faces which requires that either they or their

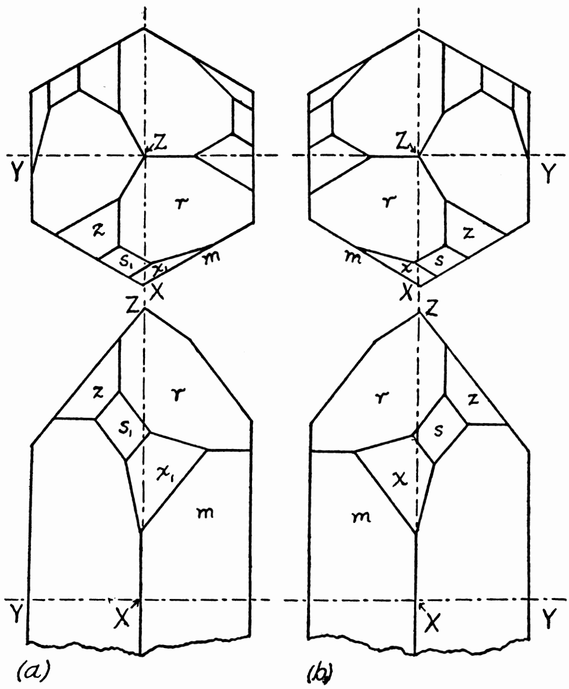


Fig. 6. (Modified from Cady and Van Dyke.) Plans and elevations of right quartz (b) (forms r $\{10\bar{1}1\}$, z $\{01\bar{1}1\}$, m $\{10\bar{1}0\}$, s $\{11\bar{2}1\}$, and x $\{51\bar{6}1\}$) and left quartz (a) (forms r $\{10\bar{1}\bar{1}\}$, z $\{01\bar{1}\bar{1}\}$, m $\{10\bar{1}0\}$, s_1 $\{2\bar{1}\bar{1}1\}$, and x_1 $\{6\bar{1}\bar{5}1\}$) showing the physical axes of reference, X, Y, and Z.

plan projections must be at angles of 60° and preferably at angles of 120° to each other.

The Schrauf method of treating crystals of the hexagonal system does not seem to be very well known, although he elaborates upon it in a book published not long after his article appeared(13). My first acquaintance with orthohexagonal axes of reference was a brief reference in a book by Bauerman (14) of the Royal School of Mines in London, an excellent textbook that came into my hands many years ago.

THE DRAWING OF ENANTIOMORPHOUS PAIRS OF CRYSTALS.

The inspiration for writing the present paper came from a perusal of an article by Cady and Van Dyke(15). In their paper they recommend the use of a right-handed axial system (physical axes of reference) for right quartz and a left-handed axial system for left quartz. Their proposal for what they call the *Right-Left axial system* is well illustrated by their Fig.

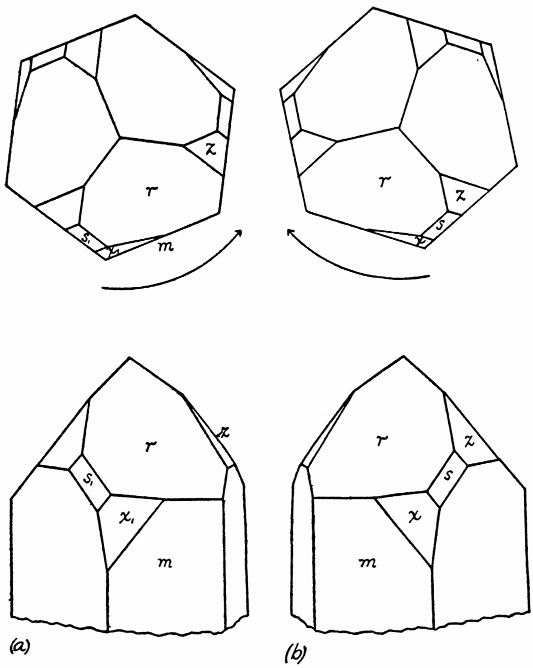


Fig. 7. Plans and clinographic drawings of right quartz (b) and left quartz (a) drawn to bring out the enantiomorphous relation of the two crystals. (Forms as given in the caption of Fig. 6.)

2 which shows combined plans and elevations drawn so that the enantiomorphous relation of the two kinds of quartz is brought out. Fig. 6 of this paper is a simplification of their Fig. 2. It is evident that the two crystals so represented are mirror-images of each other.

In 1912 the writer(16) showed the enantiomorphous relation of right and left quartz by combined plans and clinographic drawings constructed so that the rotation of the left quartz is to the right (clockwise) instead of to the left (counter-clockwise) as ordinarily. The figure of the left quartz may be very simply made by tracing the drawing of the right quartz through from the opposite side of the drawing sheet. Fig. 7 is a new drawing of a right quartz crystal and a left quartz crystal made in this way.

THE REMARKABLE PROPERTIES OF QUARTZ.

The English author Robert Greene in his fantastic tale "Orpharion," written in 1588, in the passage ". . . as full of excellent qualities as the precious stone Silex is full of secret vertue . . ." evidently refers to quartz or rock crystal.

And now at the risk of being accused of *crystallolatry* I venture to call attention to some of the remarkable properties of quartz, which is the best known and one of the most important of all minerals.

(1) The optical activity or ability to rotate the plane of polarization, so prominent in quartz, is possessed by but one other mineral (cinnabar).

(2) The elastic properties may be represented by the figure of elasticity, a dimpled rhombohedron with much-rounded edges and corners (2, p. 250).

(3) The piezo-electric properties discovered by the French physicists, Jacques and Pierre Curie, in 1880, account for the use of untwinned quartz plates as frequency controls of radio transmitters. The piezo-electric surface is a triple-almond-shaped surface, the symmetry of the entire figure being that of the ditrigonal dipyramidal class (1, p. 863; 2, p. 216).

These and other physical properties are related to the vectorially discontinuous properties of form and solution.

The symmetry formula of α-quartz (low quartz) is $A_3.3A_2$. The axis of 3-fold symmetry is the optic axis and the axes of 2-fold symmetry which are polar are the electric axes.

The symmetry, not evident in a crystal such as represented in Figure 1, is manifest in some crystals by the presence of faces of the general form $\{hk\bar{i}l\}$ and $\{i\bar{k}\bar{h}l\}$, trigonal trapezohedrons, the most common of which are $x\{51\bar{6}1\}$ and $x_1\{6\bar{1}\bar{5}1\}$, or by faces of the trigonal dipyramid $\{h.h.\overline{2h}.l\}$ and $\{2h.\bar{h}.\bar{h}.l\}$, the most common of which are $s\{11\bar{2}1\}$ and $s_1\{2\bar{1}\bar{1}1\}$ (see Fig. 7).

The symmetry is also revealed by etching either euhedral crystals such as represented in Fig. 1, oriented plates, or cut spheres.

Quartz is truly one of the most remarkable of all substances. An adequate account of its properties will be found in the excellent monograph of Sosman(17).

In conclusion I wish to record my obligation to my colleague, Professor Stephen P. Timoshenko of the Mechanical Engineering department, a former student of Voigt at the University of Göttingen, who was good enough to discuss with me some of the points brought out in this article.

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ERRATA

"PHYSICAL AXES OF REFERENCES AND GEOMETRICAL AXES OF REFERENCES FOR QUARTZ," by Austin F. Rogers. Vol. 243, July, 1945, pp. 384-392.

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384	12	"$\sqrt{3:1}:C$"	"$\sqrt{3}:1:C$"
384	24	"(clockwise)"	"(counter-clockwise)"
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387	29	"mach, Alsatian"	"mach (11), Alsatian"
390	16	"(clockwise)"	"(counter-clockwise)"
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