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HYDRAULIC GEOMETRY AND CHANNEL ROUGHNESS—A NON-LINEAR SYSTEM

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ABSTRACT. Some problems inherent in using linear statistics in hydraulic geometry studies and in developing arguments from depth and velocity to roughness rather than in the more logical, but more problematic, reverse direction are examined in the light of a reanalysis of some data from the literature, together with data provided by some English river authorities.

It is found that some relaxation of the rigidity of the power function model is desirable, and that non-linear variations of roughness may give rise to similar non-linear changes of depth and velocity with discharge.

INTRODUCTION

The consideration of roughness in hydraulic geometry investigations has always presented a problem because of the lack of objective methods of assessing Manning's "n" or the Darcy-Weisbach "f" from information about bed material and channel geometry in the broad sense, that is, including the effect of irregularities of cross section and long profile, the cause of quite considerable increases in resistance to flow (Martinec, 1972).

One attempt to circumvent this problem is expressed by the relationship between friction factor and relative roughness employed by Leopold, Wolman, and Miller (1964);

$$\frac{1}{\sqrt{f}} = 2 \log \left(\frac{d}{D_{84}} \right) + 1.0 \tag{1}$$

which is similar to the relationship derived theoretically from a consideration of the two-dimensional velocity profile (Boyer, 1954);

$$\bar{v} = 2.5 \sqrt{gRs} \log_e \left(\frac{d}{ey} \right) \tag{2}$$

where $\bar{v}$ is mean velocity, g is acceleration due to gravity, R is hydraulic radius, s is slope of the energy grade line, d is depth, e is the base of Napierian logarithms, and y is given by $y = k/30$, k being the Nikuradse sand roughness (Raudkivi, 1967, p. 223). Equating (2) with the right hand side of the Chezy formula, and substituting

$$C = \frac{\sqrt{8g}}{\sqrt{f}}$$

gives

$$\frac{\sqrt{8gRs}}{\sqrt{f}} = 2.5 \sqrt{gRs} \log_e \left(\frac{30d}{ek} \right)$$

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which reduces to

$$\frac{1}{\sqrt{f}} = 1.87 \log_{10}\left(\frac{d}{k}\right) + 0.925 \qquad (3)$$

It is surprising that this function should be fitted by stream data, since it is derived from two-dimensional considerations. Generally, Manning's "n" calculated from relative roughness alone is considerably less than that calculated from the observed velocity, depth, and slope data (Martinec, 1972). This is because the total roughness is a function of "internal distortion resistance" and "spill resistance" as well as "skin resistance" (Leopold, Wolman, and Miller, 1964, p. 162), which is effectively the relative roughness. Furthermore, the effect of transport of sediment on resistance is considerable (Vanoni and Nomicos, 1960), and the artificial Nikuradse sand roughness is inadequate when the boundary is mobile and bedforms develop. Thus the above function cannot be regarded as an entirely suitable one for calculating roughness values.

The analysis of the behavior of roughness with changes in discharge is illogical in the sense that it begins with the more readily-available data on depth and velocity and uses these to calculate the roughness measure, which is then used as a parameter (Wolman, 1955). This method is employed by Leopold, Wolman, and Miller (1964, p. 222) where the exponent of the relationship between the friction factor and discharge is derived from a consideration of "f" and "m", the depth and velocity exponents, respectively.

It is clear that this method of study has developed because of the immense problems involved in the analysis of changes of bedform (and thence, roughness) with increasing slope, depth, and discharge. However, it is because of the need to avoid these problems that other deficiencies have crept into hydraulic geometry studies. The basically empirical nature of the depth, velocity, and discharge data makes it impossible initially to infer anything more sophisticated than a simple power function relationship;

$$d \alpha Q^f,\ v \alpha Q^m,\ w \alpha Q^b, \text{ et cetera.}$$

This type of analysis has persisted because of its simplicity and because the exponents "b", "f", and "m" sum to 1 in the simplest of the continuity equation, expressed in exponent terms. In fact the development by Langbein (1964) of a statistical method for predicting the mean values of the exponents under a variety of initial constraints rests largely on the assumption that power functions (that is, relationships that are linear in logarithms) are a suitable expression of the relationships of all dependent variables to discharge.

In a sense the at-a-station hydraulic geometry has achieved a position in stream studies analogous to that of the Law of Constancy of Slopes in slope studies (Strahler, 1950). It is generally assumed that width, depth, velocity, et cetera are distributed with low standard error about a least-squares fitted regression line, invariably a power function, and that all independent variables are controlled at a given section, except discharge. In other words, the same stringent environmental limitations are imposed on the system as is the case in the Law of Constancy of Slopes.

However, the point that appears to have been overlooked is that there is no *a priori* reason why power functions should necessarily represent the relationship between the dependent variables and the discharge. Some references in the literature have indicated that simple relationships such as these are not always useful, but they have not been followed up in detail. Dawdy (1961) describes discontinuities in depth-discharge relationships following the transition from lower to upper regime, and Simons, Richardson, and Albertson (1961) state that the variations of bedforms with discharge will affect the stage-discharge relationship. Teleki (1972) claims, perhaps a little severely, that there is no convincing evidence for these sub-linear relationships because of the scatter on log-log plots. Leopold and others (1960) have investigated the nature of flow resistance in sinuous channels, and they show that a "discontinuous increase in resistance occurs" which is defined by the mean Froude number: the threshold at which the departure from the "square-law" resistance occurs may be at as low a (mean) Froude number as 0.4. Lewis (1966) notes that width does not always vary simply with discharge, at the lower end of the discharge range as well as near bankfull stage. In a later paper, however, he states that "the form of adjustments expressed by the three power functions for the Manati basin agrees with the findings of other studies conducted in widely diverse physical environments. This similarity in the orderly adjustment of river channels to discharge fluctuations indicates that these geometric and kinematic properties of streams are continuously in equilibrium." (1969, p. 293). It is hoped that the evidence presented here will suggest that the equilibrium between geometric and kinematic properties, as recognized by Church (1967), is no less continuous when it is realized that the changes of roughness with discharge are not necessarily simple power functions, and that these control the changes in velocity and depth to an extent that requires more emphasis than existing methods and concepts permit.

ANALYSIS OF FIELD DATA

Published data on occasions exhibit departures from the simple power function relationship, and it has proved possible to fit higher order curves to the data with a good deal of success. There is the obvious problem, however, that the fitting of a second order polynomial requires more data than a first order one, and in some examples sufficient data are not available.

The data tested here relate to Seneca Creek (Leopold, Wolman, and Miller, 1964, p. 216), Seven Springs and South Downington on the Brandywine (Wolman, 1955), Bernalillo on the Rio Grande (Scheidegger and Langbein, 1946), and Domingo on the Rio Galisteo (Leopold and Miller, 1956). The results of these tests are tabulated in table 1, with the results of similar curve-fitting exercises for data from several gauging stations in England. The American data are all expressed in Fps units (cubic feet per second, et cetera), and the English data in metric units (cumecs), except for that relating to the Restormel section, which are also in Fps units. The increase in explained variance or mean square due to the regression attributable to the addition of the second and third order

TABLE 1

Curve fitting to hydraulic geometry data

Gauging station	Variable	Sample size	1st degree S.S.	2nd degree S.S.	3rd degree S.S.
Seneca Creek	Depth	46	2.764	2.862**	2.887 N.S.
	Velocity	46	0.878	1.002**	1.007 N.S.
Seven Springs	Depth	10	0.747	0.751*	0.751 N.S.
	Velocity	10	0.899	0.907**	0.907 N.S.
South Downington	Depth	11	0.540	0.544**	0.544 N.S.
	Velocity	11	1.423	1.428††	1.429 N.S.
Rio Grande, Bernalillo	Depth	16	0.444	0.453**	0.454 N.S.
	Velocity	16	0.348	0.352†	0.352 N.S.
	$^{d}/_{v^2}$	16	0.265	0.309**	0.318 N.S.
Rio Galisteo, Domingo	Depth	27	2.802	3.139**	3.149 N.S.
	Velocity	27	2.108	2.136 N.S.	2.137 N.S.
	Width	27	6.975	7.420**	7.442 N.S.
R. Fowey, Restormel, Cornwall	Depth	51	1.218	1.233**	1.233 N.S.
	Velocity	51	2.419	2.424 N.S.	2.427 N.S.
R. Ryton, Serlby Park, Nottinghamshire	Width	30	0.028	0.028 N.S.	0.028 N.S.
	Depth	30	0.419	0.492**	0.494 N.S.
	Velocity	30	1.126	1.204**	1.207 N.S.
R. Derwent, Draycott, Derbyshire	Width	31	0.086	0.086 N.S.	0.087 N.S.
	Depth	31	0.643	0.717**	0.719 N.S.
	Velocity	31	0.860	0.925**	0.926 N.S.
R. Trent, Yoxall, Staffordshire	Width	24	0.008	0.011**	0.011 N.S.
	Depth	24	0.355	0.357 N.S.	0.359 N.S.
	Velocity	24	0.917	0.926**	0.927 N.S.
R. Severn, Bewdley, Worcestershire	Width	20	0.0041	0.0044**	0.0044 N.S.
	Depth	20	0.274	0.279**	0.279 N.S.
	Velocity	20	1.251	1.258**	1.258 N.S.
R. Churnet, Rocester, Staffordshire	Width	32	0.027	0.028**	0.029 N.S.
	Depth	32	1.920	1.943**	1.944 N.S.
	Velocity	32	0.432	0.422**	0.422 N.S.

Significance levels:
† significant at $p = 0.10$
†† significant at $p = 0.05$
* significant at $p = 0.025$
** significant at $p = 0.01$
N.S. not significant

terms are tested for significance using the method suggested by Chayes (1970). The data were transformed logarithmically before analysis, and the first-order curves are therefore the simple power functions. Figure 1 shows two American examples, with the power functions and the quadratic curves superimposed on the scatter diagrams, and figure 2 indicates the best fit quadratic curves for the River Ryton at Serlby Park.

The increase in explained variance due to the addition of the second order term is, in the majority of these cases, significant at the $p = 0.05$ significance level and at the $p = 0.01$ level in 21 of the 29 sets of data tested.

The sense of curvature in the relationships of depth and velocity to discharge is shown to be fairly consistent. The depth-discharge relationships are concave upward, whereas the velocity-discharge curves are concave downward. This is to be expected, in so far as, if one assumes that the width-discharge curve is adaquately described by a power function, then an increase in the slope of the depth-discharge curve must be balanced by a decrease in that of the velocity-discharge curve (or vice-versa) in order to maintain continuity. The power function curve may be expressed thus;

$$(\log Y) = \text{constant} + M\ (\log Q) \tag{4}$$

where Y is the dependent variable (that is, width (w), depth (d), or velocity (v)), Q is the discharge, and M is the exponent of the regression line. This can be generalized to include the case of the quadratic curve in logs, and three equations are produced:

$$(\log w) = b_1 + b_2(\log Q) + b_3(\log Q)^2 \tag{5}$$

$$(\log d) = f_1 + f_2(\log Q) + b_3(\log Q)^2 \tag{6}$$

$$(\log v) = m_1 + m_2(\log Q) + m_3(\log Q)^2 \tag{7}$$

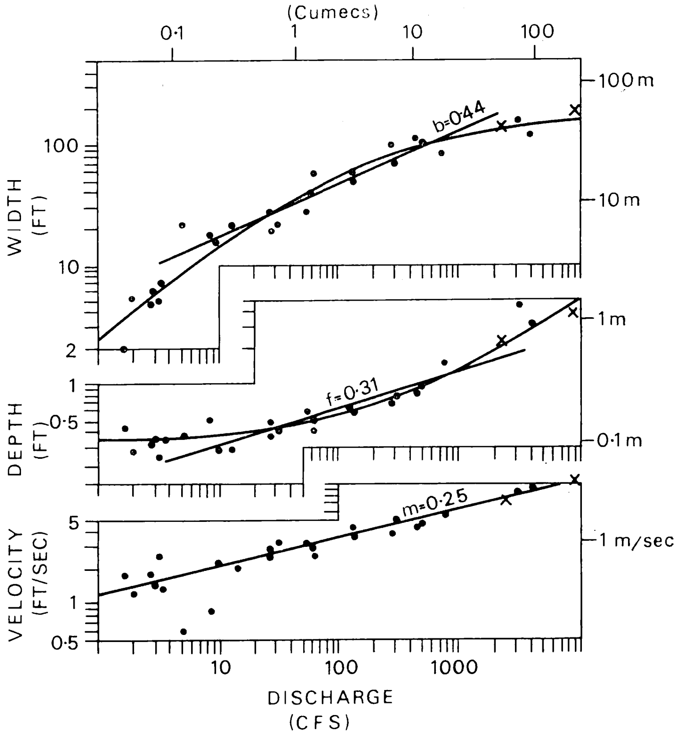
These reduce to the form of the power function when the second order term coefficients are zero. The exponent M in (4) is the first differential of the expression; so by taking the first differentials of the expressions (5), (6), and (7), the more general relation of (8) can be produced by adding these;

$$\frac{d(\log w)}{d(\log Q)} + \frac{d(\log d)}{d(\log Q)} + \frac{d(\log v)}{d(\log Q)} =$$

$$(b_2 + f_2 + m_2) + 2(b_3 + f_3 + m_3)\ (\log Q) = 1.0 \tag{8}$$

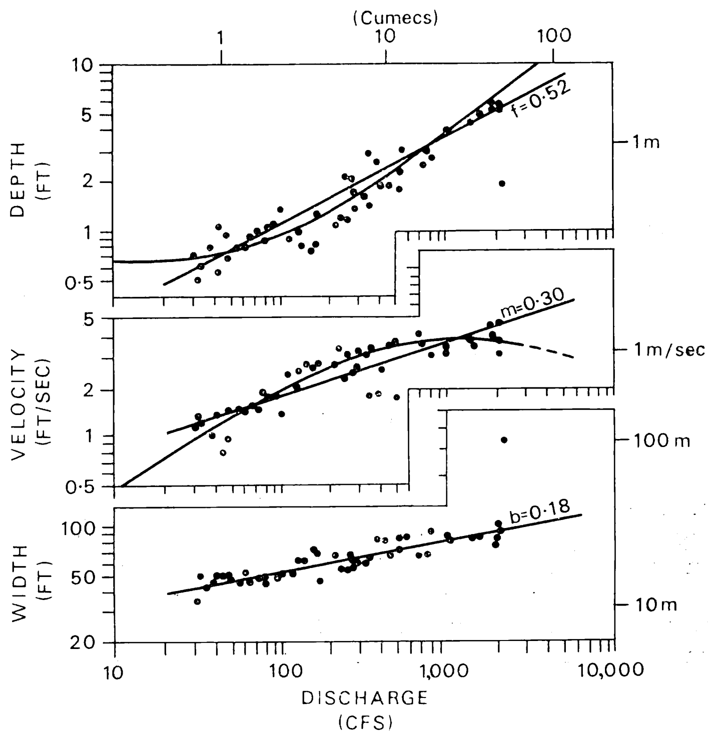
Again it can be seen that, for cases where the quadratic terms do not add significantly to the explained variance, this reduces to the well-known equation $b + f + m = 1.0$. Table 2 includes the coefficients and sums of differentials for the stations tested in this paper. In the American examples, except for Domingo, New Mex., the width data was not used, and the width-discharge curve is assumed to be a power function. The

A.



(Cumecs)
0·1
1
10
100
100 m
b=0·44
100
10 m
WIDTH (FT)
10
2
1 m
1
f=0·31
0·5
DEPTH (FT)
0·1 m
m=0·25
5
1 m/sec
VELOCITY (FT/SEC)
1
0·5
10
100
1000
DISCHARGE (CFS)

B.



(Cumecs)
1
10
100
10
5
f=0·52
1 m
DEPTH (FT)
2
1
0·5
5
m=0·30
1 m/sec
VELOCITY (FT/SEC)
2
1
100 m
0·5
b=0·18
100
WIDTH (FT)
50
10 m
20
10
100
1,000
10,000
DISCHARGE (CFS)

values substituted for log Q depend on the range of discharges occurring at a given cross section. Thus when the range is from about 1000 to 10,000 cfs, the values substituted are 3.0 and 4.0 (for example, Bernalillo), whereas for a range of 1.0 to 10.0 cumecs the substitutions would be 0.0 and 1.0 (for example, the R. Ryton at Serlby Park).

The sums of coefficients are all close to 1.0, and in general the differences between the sums for successive discharges are not significant, which suggests that the assumption that width varies as a power function of discharge is reasonable where it is made. It may be argued, however, that the Restormel and Bernalillo examples show sufficient differences between the two values for there to be the possibility of a more complex variation of width. This is discussed further below.

The association between changes in slope of the regression lines is relatively simple to account for, but the precise reasons for the upward concavity of the depth curve and the downward concavity of the velocity curve are less immediately apparent. It is proposed, however, that the rate of increase of velocity with discharge is reduced when the rate of decrease of roughness becomes less, or when roughness increases. This is evident in figure 3, which represents a plot of values of $^d/v^2$, for the Bernalillo section at which slope is constant over a wide range of discharges (Scheidegger and Langbein, 1966; and Nordin and Beverage, 1965). This, therefore, represents a plot of the way in which the Darcy-Weisbach friction factor changes with discharge, since "g" and "s" are both constants. It can be seen that roughness is here fitted by a quadratic in logarithms (the second order term increases the sum of squares explained by an amount significant at $p = 0.01$). The lack of data is unfortunate, but it is worth examining some of the comments made by Culbertson and Dawdy (1964) in a study of the Bernalillo reach. They discovered marked discontinuities in the plots of velocity against hydraulic radius, and these occur at discharges of about 3000 to 4000 cfs (85 to 113 cumecs). The Chezy C, which they use as a roughness measure, becomes constant above this discontinuity, which accords with the observation that the polynomial in figure 3 approaches its minimum within this range of high discharges. The discontinuity is a result of the change from lower to upper regimes of flow, with the attendant changes in bedform, roughness, and sediment transport.

In some cases, however, the curvature in the hydraulic geometry relationships may result from factors other than the variation of rough-

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Fig. 1.A. Relation of width, depth, and velocity to discharge at the Domingo, New Mex., gauging station on the Rio Galisteo (crosses indicate slope-area measurement) (from Leopold and Miller, 1956). Width: best-fit curve $(\log w) = 0.38 + 0.89(\log Q) - 0.11(\log Q)^2$. Depth: best-fit curve $(\log d = 0.46 - 0.09(\log Q) + 0.10(\log Q)^2$. Velocity: best-fit curve $(\log v) = 0.06 + 0.25(\log Q)$.

B. Changes of width, mean depth, and mean velocity with discharge at a river cross section, Seneca Creek, Dawsonville (from Leopold, Wolman, and Miller, 1964). (The data point at $Q = 2100$ cfs (59 cumecs) is omitted as it appears to be above bankfull). Depth: best-fit curve $(\log d) = 0.12 - 0.51(\log Q) + 0.22(\log Q)^2$. Velocity: best-fit curve $(\log v) = -1.49 + 1.36(\log Q) - 0.23(\log Q)^2$.

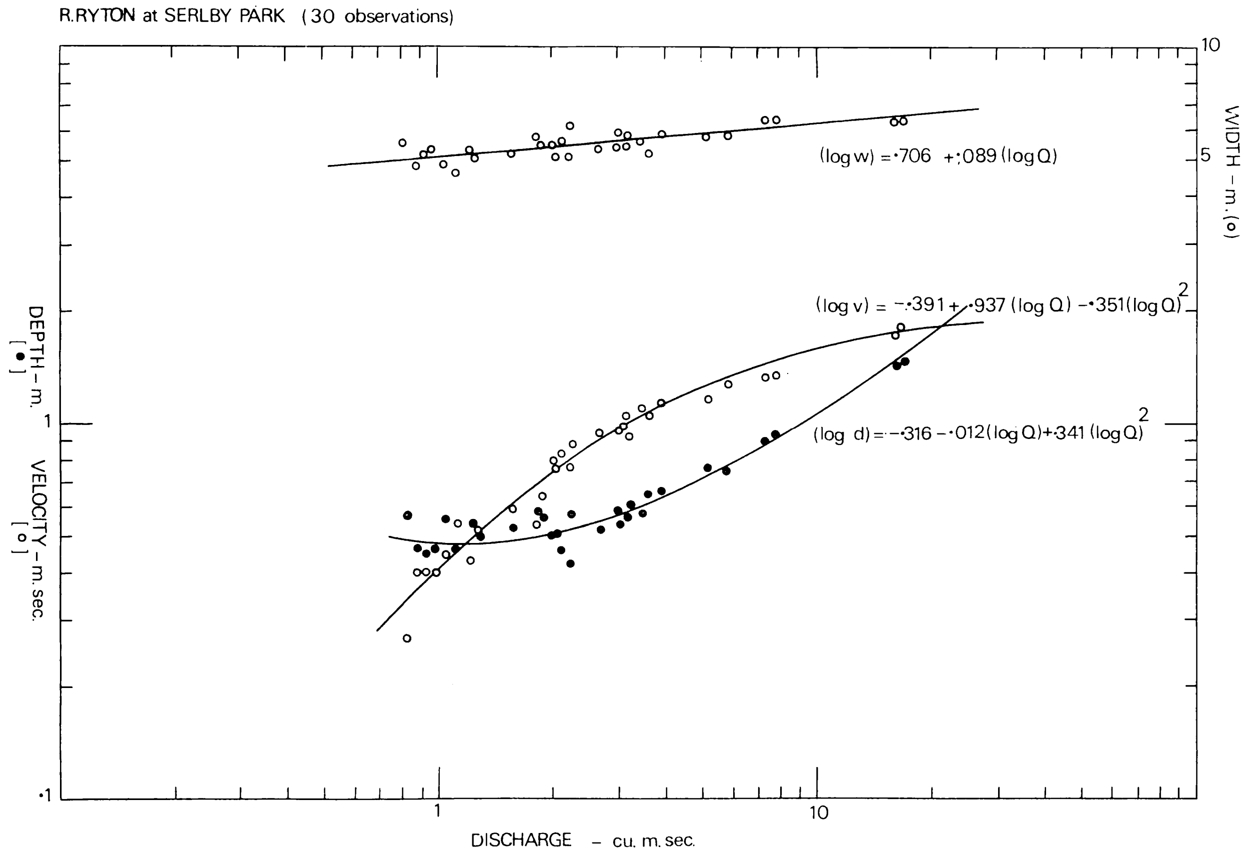

Fig. 2. R. Ryton at Serlby Park, Nottinghamshire, England. (log W) = 0.71 + 0.09(log Q); (log d) = −0.32 − 0.01 (log Q) + 0.34(log Q)^2; (log v) = −0.39 + 0.94(log Q) −0.35(log Q)^2.

TABLE 2

Sums of coefficients

Gauging station	Coefficients b_2	f_2	m_2	b_3	f_3	m_3	Sums: $(b_2 + f_2 + m_2) + 2(\log Q)(b_3 + f_3 + m_3)$ $\log Q = 0$	$\log Q = 1$	$\log Q = 2$	$\log Q = 3$	$\log Q = 4$
Seneca Creek	0.22	–0.51	1.35	0.00	0.22	–0.23	—	—	1.02	1.00	—
Seven Springs	0.05	0.17	0.89	0.00	0.07	–0.09	—	—	1.03	0.99	—
South Downington	0.04	–0.00	1.01	0.00	0.08	–0.09	—	—	1.01	0.99	—
Bernalillo	0.00	1.64	–1.26	0.00	0.25	–0.17	—	—	—	0.91	1.09
Domingo	0.90	–0.09	0.25	–0.11	0.10	0.00	—	—	1.01	0.98	0.97
Restormel	0.02	–0.12	0.89	0.00	0.12	–0.07	—	—	0.97	1.06	—
Serlby Park	0.09	–0.01	0.94	0.00	0.34	–0.35	1.02	0.99	—	—	—
Draycott	0.15	–0.43	1.24	0.00	0.31	–0.29	—	0.99	1.02	—	—
Yoxall	–0.27	0.36	1.16	0.11	0.00	–0.21	—	1.08	0.88	—	—
Bewdley	0.05	–0.10	1.15	0.03	0.13	–0.16	—	1.00	1.00	—	—
Rocester	0.15	0.35	0.48	–0.06	0.22	–0.15	0.98	1.01	—	—	—

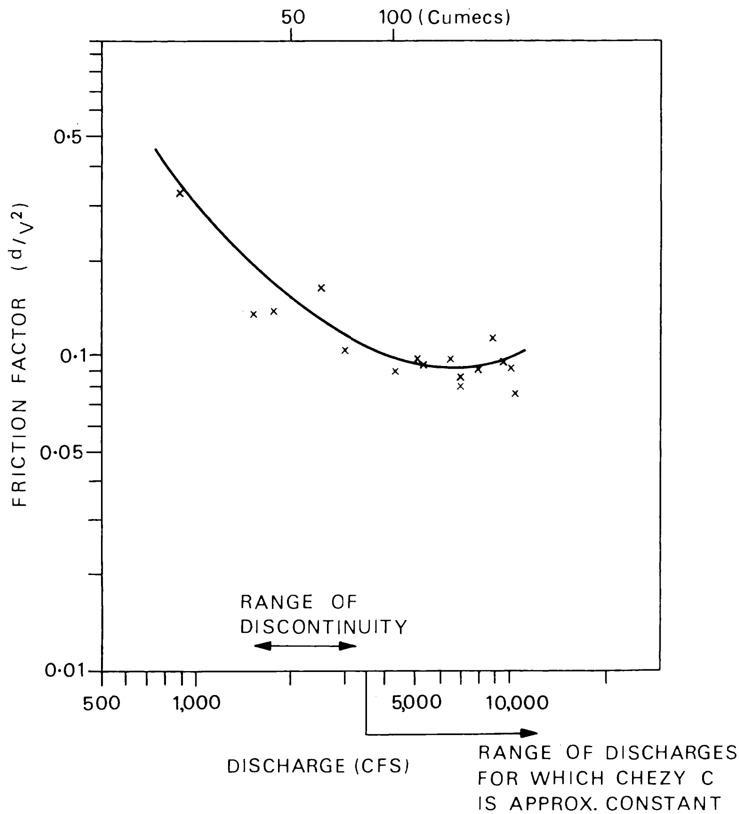

Fig. 3. Friction factor changes with discharge, Bernalillo, Rio Grande. Best-fit curve: $(\log^{d}/v^{2}) = 7.71 - 4.51\ (\log Q) + 0.58\ (\log Q)^{2}$.

ness associated with change of regime. The depth-discharge relationship is curved in the case of the Restormel data, but the velocity curve is not significantly quadratic. The reason for this may well be that the behavior of width at this section is not as simple as has been assumed. In fact, the section at Restormel is partly artificial, in that to simplify the gauging procedure brick walls have been built 17 m apart. At low discharges there is a gradual but slow increase in width, but above 70 to 80 cfs (2 to 2.3 cumecs) the width is constrained. At these higher discharges the width is therefore less than expected and the depth is greater than expected by way of compensation: thus the depth-discharge curve increases in slope. The Rio Galisteo at Domingo exhibits the same kind of effect, the width and depth curves being quadratic in logarithms, rather than simple power functions. Three of the examples from England illustrate this further. The Severn at Bewdley and the Churnet at Rocester have quadratic width, depth, and velocity curves, whereas the Trent at Yoxall has a depth-discharge relationship which is a power function, but which may tend toward a quadratic relationship as well (compare the large change in the sums of the coefficients in this case). It would appear that any tendency of the width to vary in a non-linear manner below bankfull stage also affects the other curves. In effect the cross-sectional roughness is varying irregularly, and changes in the hydraulic radius are transmitted to the velocity through the roughness. Furthermore, a consideration of the implications of the work done by Leopold and others (1960) leads to the conclusion that it is unlikely that roughness should change in a simple fashion as discharge increases, because although the increasing depth and hydraulic radius cause a steady reduction in resistance to flow, the Froude number usually increases at the same time; thus the onset of "spill resistance" occurs, and an additional element must be included in the total roughness. The rate of change of roughness is thus fairly complex and is by no means necessarily linear.

It is evident that a close relationship must exist between roughness and velocity, as is expressed in the Manning equation. If one assumes a channel with constant width and slope (for example, the Rio Grande at Bernalillo or the laboratory flume) and depth and velocity exponents "f" and "m" respectively, then the exponent in the relationship between the Darcy-Weisbach friction factor "*f*" and discharge is given by f-2m ($^{d}/v^{2}$ in exponent terms). Thus a very rapid increase in velocity with increasing discharge (m >0.5) is associated with a rapid decrease in "*f*" (because 2 m >> f).

This, of course, requires generalization for the case where slope is not constant. For instance, the different water surface slope at riffle and pool locations makes it impossible to assume constant slope. Table 3 illustrates the effect of varying slope using data collected for the River Fowey in Cornwall. The fifth column is the exponent expected from consideration of the exponents of depth (f), velocity (m), and slope (z),

TABLE 3

Relations between velocity, slope, and friction

f	m	z	f − 2m	f + z − 2m	Friction factor exponent (least-squares)
0.48	0.50	−0.04	−0.52	−0.56	−0.59
0.21	0.78	1.51	−1.35	0.16	0.16
0.44	0.51	−0.31	−0.58	−0.89	−0.89
0.15	0.77	2.01	−1.39	0.62	0.64
0.12	0.59	−0.30	−1.06	−1.36	−1.37
0.44	0.49	−0.06	−0.54	−0.60	−0.58

f = depth exponent
m = velocity exponent
z = slope exponent

It should be pointed out that these data apply over a relatively small range of discharge, well below bankfull stage. Best fit curves, therefore, took the form of power functions in these examples, and the development of simple expressions in terms of exponents was possible.

while the sixth column contains the exponents of the least-squares regressions of $^{ds}/v^2$ against Q. The similarity between these values is encouraging in view of the variance inherent in field data collection, particularly in the measurement of water surface slope. From the table it is evident that when a high value of "m", that is, a rapid increase of velocity, does not occur in conjunction with a highly negative value of the friction factor exponent, it is because the slope is increasing rapidly with discharge—that is, in a pool location, in a stream characterized by alternating riffles and pools. Clearly the variation of slope will have a considerable influence on the rate of change of the friction factor, and the assumption that slope is constant which is usually made in a minimum variance solution for the at-a-station exponents (see table 4 in Langbein, 1965) is rather questionable.

There are obvious discontinuities in hydraulic geometry relationships at bankfull stage, but it is also apparent from the evidence presented here that non-linear responses by the dependent variables to a change in the independent variable, discharge, can also occur at discharges below bankfull. The power function is therefore not a general enough empirical model, and it would seem empirically reasonable to relax the rather rigid nature of this model and replace it by the more general polynomial in logarithms, which allows the considerations of non-linear variations of width, depth, velocity, and roughness and reduces to the power function when the coefficients of the terms higher than first order are zero. The discontinuities noted in bedform experiments as Froude numbers increase and in the behavior of width (Lewis, 1966) provide two reasons for the necessity of a more generalized model. The acceptance of a higher order curve as such a generalization is therefore a possibility; certainly it would seem to enable the continued use of simple continuity relations between width, depth, and velocity coefficients. There are some problems still worth consideration, however. To begin with, a quadratic in logs may not seem to be a simple relationship to deal with, even though equation (8) does not present much difficulty. Attempts to use untransformed data were not successful, however, because generally higher-order terms (particularly the third-order term) were also significant, because it was possible to obtain least-squares fitting to the very widely-spaced data points in the high discharge range. The log-transform produces better conditions for fitting polynomials, because the data are more nearly equally spaced; thus the log-normal nature of the distributions of most of the variables involved favors the use of logarithms, and the inflections obtained by fitting third-order curves to untransformed data were regarded as spurious.

Similar restraints exist in the use of second-degree equations as apply to the simpler power functions, although they present greater pitfalls to the unwary. The problem of extrapolation is greater, and there is no justification for continuing the parabola of figure 3, for instance, as there is nothing in the data to suggest that friction will increase at higher discharge still. The same applies to the velocity curve in figure 1B, which reaches its peak before bankfull stage. The implication of this is not that at discharges in excess of 1000 cfs (28 cumecs) the velocity decreases, but that a period of fluctuating velocity occurs when, statistically speaking, there is no significant change in velocity. Since it is often in this range of discharges that the least frequent gaugings are made, an increase in the number of sample points may shift or broaden the peak of this curve and reduce the extrapolation problems. It should be noted, perhaps, that there is a possibility that a better fit than is the case with the power function may be achieved by regressing the untransformed dependent variable against the logarithm of discharge ($y = f(\log Q)$). This would, however, lose the advantage held by the polynomial of simple reduction to the power function when the second order coefficients are zero. Finally, one other problem is that the unexplained variance due to experimental error is often greater in the velocity-discharge relationships than in the depth and width ones; consequently it is less easy to detect small amounts of curvature empirically.

Since there are likely to be more marked discontinuities in the hydraulic geometry of flumes than in that of natural streams, it is to be expected that further evidence of non-linearities may be provided by an analysis of data collected in flumes. Langbein (1964) appears to believe that the assumptions of the minimum variance model will still hold and derives the predicted exponents for recirculating and non-recirculating flumes. Some preliminary results indicate that considerable departures from the model may occur. Although the best-fit power functions for a set of at-a-station data (collected in the Cambridge Geography Department flume) produce exponents of $f = 0.574$ and $m = 0.425$—compared to 0.58 and 0.42 predicted for similar conditions by Langbein—plots of depth and velocity against discharge indicate that simple power functions are only approximations. Velocity increases with discharge less rapidly in the range of discharges in which ripples develop to maximum amplitude and thus cause an increasing resistance to flow. The discontinuous changes of roughness cause non-linear responses of depth and velocity, and the extent to which depth and velocity absorb the effect of an increase in discharge will depend on the behavior of roughness. In natural streams the thresholds are less well defined, and continuous curves best fit the behavior of the variables involved, however.

STATISTICAL IMPLICATIONS

It is interesting to consider the implications of the foregoing discussion to the statistical use of the power function model, particularly with regard to inferential problems. Although the hydraulic geometry regressions are often only used in a descriptive manner, it is frequently desirable to infer from them certain features of the system under study. For example, it may be of interest to be able to state that depth increases more rapidly at a given section than velocity, or that these variables behave differently between two sections, or that the downstream behavior of one basin differs significantly from that of another. The power function model provides a simple means of doing this, through the t-test of regression slopes (Dixon and Massey, 1951, p. 208-209).

However, if, as we have seen, quadratics can often be regarded as the true relations between the two variables in these regressions, it is likely that the residuals from the linear relation will exhibit auto-correlation (Johnston, 1963, p. 177). That this is, in fact, the case can be seen from table 4. The residuals from the first order regressions for the sets of data already analyzed have been tested using the Durbin-Watson test (Johnston, 1963, p. 192) and the non-parametric Runs test (Siegel,

TABLE 4

Tests for autocorrelation of residuals from linear regression

River	Gauge	Variable	Sample size	d	z
Seneca Creek	Dawsonville	Depth	46	1.708 N.S.	−0.31 N.S.
"	"	Velocity	46	1.309*	−2.01†††
Brandywine Creek	Seven Springs	Depth	10	1.663 N.S.	—
"	"	Velocity	10	1.556 N.S.	—
"	South Downington	Depth	11	1.070††	—
"	"	Velocity	11	1.154††	—
Rio Grande	Bernalillo	Depth	16	2.176 N.S.	—
"	"	Velocity	16	1.584 N.S.	—
"	"	d/v^2	16	1.986 N.S.	—
Rio Galisteo	Domingo	Depth	27	1.751 N.S.	−0.44 N.S.
"	"	Velocity	27	0.919**	−3.69***
"	"	Width	27	0.988**	−3.73***
Fowey	Restormel	Depth	51	1.290**	−2.69***
"	"	Velocity	51	1.835 N.S.	−2.36***
Ryton	Serlby Park	Width	30	2.431 N.S.	0.86 N.S.
"	"	Depth	30	0.867**	−3.35***
"	"	Velocity	30	0.921**	−4.08***
Derwent	Draycott	Width	31	1.661 N.S.	−1.43 N.S.
"	"	Depth	31	0.230**	−4.13***
"	"	Velocity	31	0.470**	−3.42***
Trent	Yoxall	Width	24	1.054*	−2.51***
"	"	Depth	24	1.687 N.S.	−1.65 N.S.
"	"	Velocity	24	0.996**	−3.33***
Severn	Bewdley	Width	20	1.069*	−3.20***
"	"	Depth	20	0.551**	−3.67***
"	"	Velocity	20	0.372**	−3.67***
Churnet	Rocester	Width	32	1.474††	−1.77 N.S.
"	"	Depth	32	0.781**	−2.52***
"	"	Velocity	32	1.271†	−3.20***

d = Durbin-Watson statistic
** positive autocorrelation at p = 0.01
* positive autocorrelation at p = 0.025
† positive autocorrelation at p = 0.05
†† test inconclusive
N.S. no positive autocorrelation
(d between upper and lower bounds of tabulated values)
z = normal deviate for large sample runs test
*** significant at p = 0.01
††† significant at p = 0.05
N.S. not significant

1956, p. 56). It can be seen that the majority of relationships that were significantly improved by the addition of the quadratic term exhibit autocorrelation, as indicated by the Durbin-Watson statistic "d", and this is also evident in the "z" statistic (a normal deviate), which indicates that there are far fewer runs of negative and positive residuals than expected in a sequence of random, independent scores. From the point of view of this discussion, there are two main implications of the presence of autocorrelation. Firstly, there is the frequently serious underestimation of standard errors when autocorrelation is present, and secondly the numerical estimation of the parameters of the regression line will be affected. Since the depth-discharge curves consistently have positive first derivatives and the velocity-discharge curves negative ones, the patterns of residuals from power function approximations are consistent, and the effect of correcting for the presence of autocorrelation is to cause the slope values to change upward for the former and downward for the latter. The method used to remove the autocorrelation involves transforming the original variables according to the autoregressive structure of the disturbance term and applying simple least-squares to the transformed variables (Johnston, 1963, chap. 7 especially p. 197-199). In terms of at-a-station problems, there are two important results to consider, and they are summarized in figure 4. Here, the lines A-A′ are the fitted power functions, B-B′ represent the quadratic curve followed by the data, and C-C′ represent the corrected regressions. In figure 3A, the depth appears to increase more rapidly with discharge than velocity. Correcting the regressions, therefore, causes the difference between the slopes to be greater than originally predicted. In figure 3B, with $f < m$, the values of the corrected exponents converge. In conjunction with the increase in the standard error, it is evident that considerable changes may occur in the value of "t". Since

$$t = \frac{b_1 - b_2}{S_{y.x.p}\sqrt{\frac{1}{(n_1-1)S_{x1}^2} + \frac{1}{(n_2-1)S_{x2}^2}}}$$

(Dixon and Massey, 1951, p. 208-209), in the case illustrated in 3A, the numerator will increase, whereas in 3B it decreases. The denominator will increase consistently. The results of some empirical examples are given in table 5, where the "f" and "m" values of some sections are compared before and after correction for the effect of autocorrelation. Some between-section comparisons (that is, pairs of "f" or "m" values) are also tabulated. It is evident from the changes in "t" that, apart from the misleading physical interpretations that may be made when fitting power functions, there are also considerable inferential difficulties caused by the failure to recognize that the relationships may be more complex than this.

In addition to this problem, it is important to consider the implications as far as minimum variance theory is concerned (Langbein, 1964).

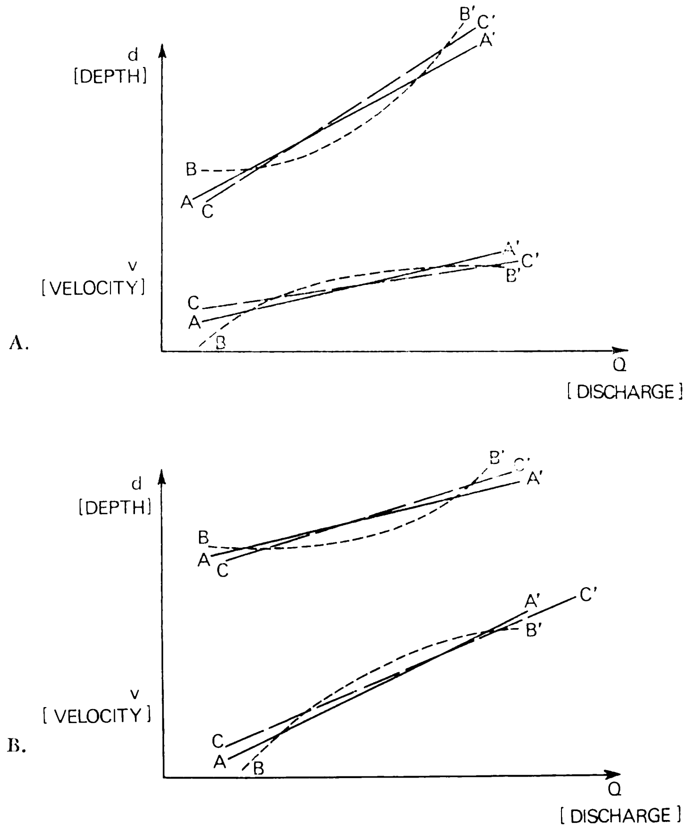

Fig. 4. Effects of autocorrelation of residuals on regression inference.
A. Depth increasing more rapidly than velocity ($f > m$).
B. Depth increasing less rapidly than velocity ($f < m$).

The critical assumption of this theoretical method of predicting exponents is that the exponent is a surrogate for the variance of the dependant variable. This assumption can be derived from linear statistics (assuming the data is transformed using logarithms). Croxton (1959), for example, provides the necessary basic relations. The regression coefficient, b_y, is defined thus:

$$b_y = \frac{\sum xy}{\sum x^2}$$

and the correlation coefficient, r, is:

$$r = \frac{\sum xy}{\sum x^2 \sum y^2}$$

Since $\sum x^2 = NS_x^2$

$$b_y = \frac{\sum xy}{NS_x^2}$$

and

$$r = \frac{\sum xy}{NS_xS_y}$$

Thus

$$b_y = r\frac{S_y}{S_x}$$

In a given problem, S_x (here the standard deviation of the independent variable, discharge) is constant, and if the correlation is assumed to be perfect ($r = 1.0$), it follows that

$$b_y \propto S_y$$

The critical problem is thus that the model assumes linearity, and it has been shown above that this assumption is not necessarily valid. The

TABLE 5

Effects of correcting regressions

Gauging station	Before correcting regression Exponents compared	t	After correcting regression Exponents compared	t
Rocester	f = 0.624 m = 0.296	t = 20.81	f = 0.639 m = 0.295	t = 8.69
Serlby Park	f = 0.346 m = 0.568	t = − 6.43	f = 0.440 m = 0.474	t = −0.50
Yoxall	f = 0.363 m = 0.583	t = −12.49	f = 0.370 m = 0.570	t = −5.07
Draycott	f = 0.396 m = 0.458	t = − 2.43	f = 0.576 m = 0.320	t = 2.87
Bewdley	f = 0.307 m = 0.655	t = −31.78	f = 0.342 m = 0.603	t = −6.81
Serlby Park Yoxall	m = 0.568 m = 0.583	t = − 0.35	m = 0.474 m = 0.570	t = −1.65
Bewdley Yoxall	m = 0.655 m = 0.583	t = 3.53	m = 0.603 m = 0.570	t = 0.62
Serlby Park Draycott	f = 0.346 f = 0.396	t = − 1.16	f = 0.440 f = 0.576	t = −1.34

above formulae for "b_y" and "r" cannot be used meaningfully if the best-fit regression is quadratic, and if they are used, and a linear approximation to the trend is calculated, "r" cannot be 1.0 and will not be a constant but will vary with the degree of curvature of the quadratic (that is, the degree of departure from linearity).

CONCLUSIONS

From the limited amount of data presented here, it would appear that the rates of change of both depth and velocity are related to that of roughness. Depth and velocity are, in fact, functions of roughness, and when the rate of change of roughness is non-uniform (see, for example, Church, 1967, p. 860, and ms, figs. 80A and 80B), this will cause departures from simple power function relationships in the case of depth and velocity. Although, in a sense, the statement that depth and velocity are dependent on roughness is hardly an advance on previous work, for the Manning equation has expressed the relationship for many years, the precise nature of the control involved has not been successfully built into hydraulic geometry investigations.

Since depth and velocity are functions of roughness, they must be functions of channel geometry and sediment characteristics—bed material in particular. As Church (1967) has pointed out, the channel geometry influences the flow geometry because the shape of the channel determines the extent to which the momentum of flow is transmitted to the walls through the turbulence of flow—that is, the influence is through the roughness of channel shape. The correlation between hydraulic radius and roughness is a negative one; that is, the greater the ratio of cross-sectional area to wetted perimeter, the smaller is the resistance provided by the channel geometry. This relationship is not the one at first suggested by inversion of the Manning Equation, where $n \alpha R^{2/3} s^{1/2}/v$. This is probably because the interdependence of "R", "s", and "v" can lead to the result that "n" varies inversely with "R" when several sections of differing geometry are sampled. Furthermore, the Manning equation originally had velocity as the dependent variable, and although not a least-squares regression, it presumably approximates to it. Consequently, inversion of the formula is not strictly valid, since the error distribution of "y" on "x" is clearly not the same as that of "x" on "y". It is not necessarily the case that the coefficients of "R" and "s" will remain respectively $\frac{2}{3}$ and $\frac{1}{2}$, if the data is reanalyzed using "n" as the dependent variable.

Achieving a point of entry into this system for the purpose of analysis is extremely difficult, since the interaction is complete. Searching for simply-directed relationships is also fraught with difficulty. Depth and velocity are functions of roughness, which is partly a function of channel geometry, which is in turn partly a function of depth and velocity. However, it appears to be the case that the neglect of the links between roughness and depth and velocity has led to some oversimplifications as a result of the tendency to concentrate on the empirically available data and infer roughness from this.

Thus the implications of the results discussed here are of some importance at more than one level. To begin with, as far as the problem of hydraulic geometry is concerned, it is evident that some link is necessary between the work of Leopold and Maddock (1953) and Wolman (1955), on the one hand, and Simons, Richardson, and Albertson (1961) on the other. Both these groups of workers have employed essentially empirical methods to tackle problems that though basically somewhat different must be interrelated. The relaxation of the rigid assumptions of the power function model may help to make it possible for this linkage to occur. Furthermore, it is probable that similar curve-fitting exercises will also be successful in the case of downstream changes of the hydraulic variables with discharge, since the downstream changes of roughness may also be non-linear. In fact, the environmental variables are uncontrolled in this case, and it is even less likely that the power function will represent a general model. The application of higher order equations is therefore possible, and the use of equation (8) in the place of the simple continuity relation $b + f + m = 1.0$ is equally valid for the downstream case.

At another level—a more general, conceptual one—the problems introduced by the assumption of linearity are considerable. In Langbein's (1964) minimum variance theory for the prediction of the mean values of exponents, the assumption that linear relationships between the variables (logarithmically transformed) is basic, and the exponents themselves are assumed to be surrogates for the variances of the dependent variables concerned. Given that the relationships are not necessarily power functions, the basis for this assumption will not be acceptable. Furthermore, it is worth considering the problems introduced to systems analysis by the conclusions mentioned here. Correlation structures of the type developed by Melton (1958) operate on the assumption that the relations between variables can be described adequately and analyzed by linear statistical techniques. In many cases these methods may be suitable, but as Pitty (1972) has noted, there are frequent examples where non-linear relationships have been found (for example, Hjulstrom, 1935; Schumm and Langbein, 1958). It is evident that care must be taken in the analysis of systems involving sediment transport, sediment yield, soil strength and characteristics, and hydraulic geometry: it is not improbable that other empirically investigated systems may exhibit non-linear relationships between their components.

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