THE PHYSICS OF UPWARD DRILLING.

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ABSTRACT. A theory of 'upward drilling' by gas-bearing solutions through a ceiling of a solid phase is developed on the basis of experiments with ice as the solid phase and various fluids and solutions as the liquid phase. It is shown that the composition of the gas phase is unimportant.

According to this theory, there is one particular size of bubble that is most active, and hence one particular size of cavity that tends to progress most rapidly, with a particular set of components. The size is calculated for ice attacked by alcohol, refractories by molten glass, and for rocks attacked by viscous magmas. It would appear that the order of magnitude is right in each case.

THE fact that downward-facing surfaces of refractory in a glass melting furnace commonly become deeply pitted has been observed for many years. Such surfaces are for instance the 'rings' and 'floaters' which act as barriers to the movement of scum and floating particles. The same is true of the lintel block of a tank throat. And because the corrosion on downward-facing surfaces is not only pitted but rapid, it became standard practice some 20 years ago to make the tank walls in a single course of blocks, so as to avoid horizontal joints, which must necessarily provide at least one downward facing surface. Where there is no means of avoiding such a surface, as in the throat of a tank, for instance, there was provided the first generally accepted location for 'super-refractories' in tank construction.

The practical importance of the matter is well attested by the design of almost every glass tank in the country, but the literature on the subject is very scanty. The mechanism that brings about the pitted conditions has been generally called 'upward drilling' and has been investigated by McCauley[1] and Ross.[2] McCauley recognized that at the top of an upward drilling cavity there was a gas bubble, and that the bubble was an essential part of the mechanism; McCauley further recognized that surface tension played an important part in this and some other forms of refractory corrosion.

As a rule the pits are not deep, because as a number of holes drill upward in relatively close proximity, the intervening

[1] McCauley, G. V.: Discussion in 'Symposium on Glass Industries Refractories', Bull. Am. Cer. Soc. 4.605, 1925.

[2] Ross, D. W.: Jr. Am. Cer. Soc. 9.641, 1926.

ridges, surrounded on all sides by solvent glass, also dissolve away, and the stable configuration seems to be one with pits whose depth is comparable with their diameter (Plate I, Fig. 1). With reasonably homogeneous blocks, all the pits have comparable depths, and one pit is not vastly deeper than another. Moreover, all the pits are of comparable diameters, about big enough to stick one's thumb into. In a glass tank we do not find pits 3″ or 4″ in diameter, nor do we find them 1/10″ or even ¼″ in diameter. They tend to be around 1″. There must, therefore, be some definite mechanism regulating the size of the bubbles responsible for the drilling, or causing a particular size of bubble to be most effective. This observation caused one of the present writers (F. W. P.) a good many years ago to begin experiments with 'models' made with transparent liquids and solids, to find out what the mechanism was.

The first 'model' proved to be very fortunate. The solid chosen was ice. The liquid was a mixture of alcohol and water adjusted to the same density as ice. The experiment was carried out below zero centigrade, in order that the ice should 'melt' only by chemical attack (solution) and not by heat. The ice was supported over a beaker with its lower, downward-facing surface immersed in the liquid. It was immediately observed that tiny bubbles formed and danced agitatedly up and down below the ice, and started to drill upward. As time went on, the bubbles increased in size, and the pits increased in diameter, till they were perhaps of the order of 1/16″ across, but they did not increase beyond this size (Plate II, Fig. 1). This seems to be the stable scale of operations for this particular combination of substances, and it is a much smaller scale than obtains with molten glass and tank block blocks. After trying various other materials for experiment, it became apparent that upward drilling in ice did not take place in the absence of bubbles, and that the size of the holes depended on the size of the bubbles, and that there was some size of bubble and hole that was most active and therefore most stable.

The possibility that 'upward drilling' on the earth's crust may be the explanation of volcanoes should not be overlooked; but a volcano vent is much larger in diameter than the 1″ holes in a tank block, and if it can be accounted for in the same fashion, the physical quantities involved must be numerically quite different. As will be shown later, there is no apparent impossibility in these three phenomena being essentially the same.

Extent of Investigation.

The experimental work herein reported consists in investigating the behavior of bubbles by direct observation, a thing which cannot be done with molten glass. The behavior of bubbles of various sizes and of different gases was examined and the various fluids were investigated to see what combination of physical properties was necessary for upward drilling to start and continue. The observations, therefore, differ considerably from those reported hitherto, which concern the appearance of refractories after removal of glass from the furnace, or after removal of refractories from the tank.

The limitations of the experimental work made it advisable to formulate a mathematical treatment, which also is here attempted. The degree of agreement between theory and observation is indicated for several cases of practical interest.

Physical Properties Involved in Upward Drilling.

Upward drilling represents an acceleration in the rate of solution of a solid, in which some of the available mechanical energy is used to increase the rate of solution. The rate of drilling depends on a number of the physical properties of the materials involved, particularly on the changes in some of these properties as the solution is diluted by the dissolving solid. Variations in surface energy of the bubble during drilling represent the most important of these; the surface of a bubble absorbs the component of a solution which lowers surface tension, and upward drilling may occur if this component helps dissolve the solid. In particular, it seems that the surface tension forces are primarily responsible for the characteristic motion of gas bubbles while they are drilling upward. Accordingly, the composition of the gas in the bubble is not of much consequence, since the nature of the gas above a liquid surface has little effect on its surface energy.

Energy Relations in Upward Drilling, Neglecting Viscosity of Solutions.

At least two forces act on a bubble moving about in a liquid of varying composition; the gravitational force, and the force caused by variations in surface energy. In a static liquid, the latter force has a potential, given by the surface energy, $\gamma\sigma$, where γ is the surface tension, and σ is the surface area. The force of gravity is ordinarily the most important one for the bubble, but it can be overbalanced by the other force which is given by the gradient of surface energy. This happens during upward drilling at the lower ice surface, for example, in the case of ice attacked by alcohol, and it is clear that there is a large gradient of alcohol concentration, and hence of surface tension, at this part of the liquid.

During upward drilling of gas bubbles in alcohol solutions on ice, small bubbles are continuously jumping downwards from their position in contact with the lower ice surface, where the concentration of alcohol is least, to some distance from the ice, where the concentration of alcohol approaches its mean value for the solution. In moving from pure water to an alcohol solution, a bubble loses surface energy $\Delta E = \Delta\gamma\sigma$, where $\Delta\gamma$ is the difference in surface tension of water and of the alcohol solution. This may appear as mechanical energy, and it could then push the bubble downwards a distance h, where mgh is the energy required, neglecting viscosity, and m is the mass of fluid displaced by the bubble. $\frac{\Delta E}{mg}$ is therefore an upper limit for h. For a solution of alcohol in water, $\Delta\gamma$ might be 30 dyne/cm., and a bubble of radius 0.1 mm. could be moved 9 cm. downwards. Small bubbles in alcohol solutions do not move as much as this, however, because of the viscosity of the solution, the effect of which will be considered later.

This calculation can be applied to upward drilling providing the change in surface energy can be used to move the bubble. Suppose the bubble is resting on the lower surface of the ice interface, where it is surrounded by nearly pure melted water. Since alcohol lowers surface tension, the bubble can release energy by absorbing alcohol. Then when a streamline of more concentrated alcohol solution happens to touch the lower surface of the bubble, alcohol will start upwards to be absorbed, and mechanical reaction will start the bubble moving. Since momentum is conserved in this system, the initial velocity of the bubble is downwards, and, from the conservation of energy, we shall later find the downward motion of the bubble in detail.

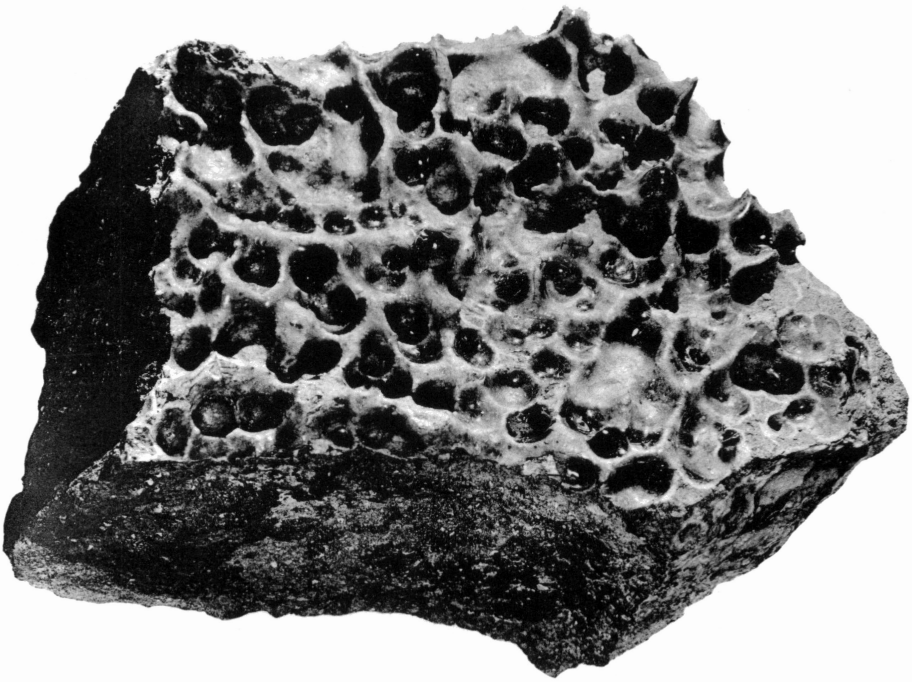
Ordinarily, a solid dissolving in a solution is surrounded by a layer of liquid different in temperature or composition from the solution itself. If a bubble is to accelerate attack on the lower face of the solid, it must carry material through this small region next to the solid surface. The surface energy of a bubble when it is immersed in the region near the solid must therefore be higher than when it is surrounded by solution some distance from the solid. The difference in surface energy, $\Delta\gamma$ is a measure of the mechanical energy which the bubble receives from surface tension forces. $\Delta\gamma$ may be estimated if the surface energy of the solutions involved is known; with ice, water, and alcohol, for example, the surface tension change is approximately the difference between the surface tension of water and that of the alcohol solution. With other kinds of solutions, however, this is not necessarily true. With wetting agents, such as sodium oleate, the maximum effect on the surface tension of water is produced by the first 0.1 per cent or so which is added, and the surface energy of a bubble might be nearly the same near the ice as some distance from it.

$\Delta\gamma$ is also a measure of the amount of material carried from the solution through the liquid layer surrounding the solid. In the case of ice dissolving in water, any substance in solution acts as an aid to solution of ice. This also is not generally true, and, during attack on other materials, surface energy might tend to bring some totally useless substance to the surface of the solid.

The physical mechanism of attack by upward drilling bubbles thus seems to be one in which the bubble, initially at rest on the downward-facing surface, picks up the material which lowers its surface energy, jumping downwards off the surface as a result. It may thus reach more concentrated solutions further from the ice. Finally, having absorbed all it can of the material which lowers surface tension, it is forced by gravity back up against the solid surface, carrying this material with it. Here it can release the material, which is diluted by its attack on the solid, and the surface energy of the bubble returns to its initial value.

The following conditions necessary for the existence of upward drilling are therefore suggested:

1) The density of solution and dissolved solid must be greater than that of the solution alone, otherwise action will stop when a relatively small amount of the solid has been attacked.
2) The surface tension of the solution must be appreciably increased by the solution of the solid.



Appearance of the downward-facing side of a refractory block, after its removal from a glass tank. The pitted condition of the surface is caused by upward drilling. A gas bubble would be found in each of the pits in this surface, before removal of the surrounding glass.

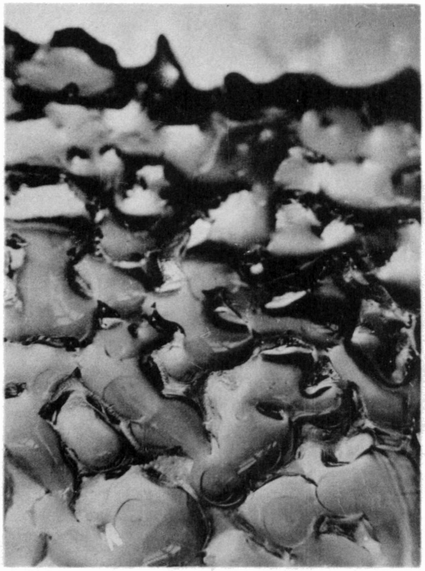


FIG. 1

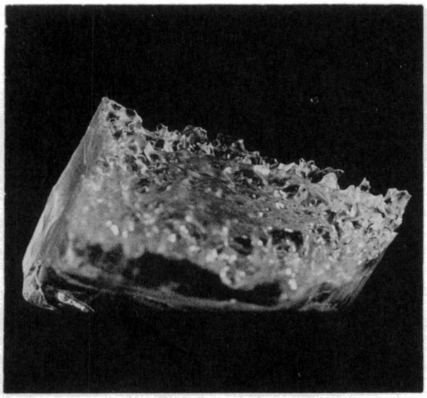


FIG. 3

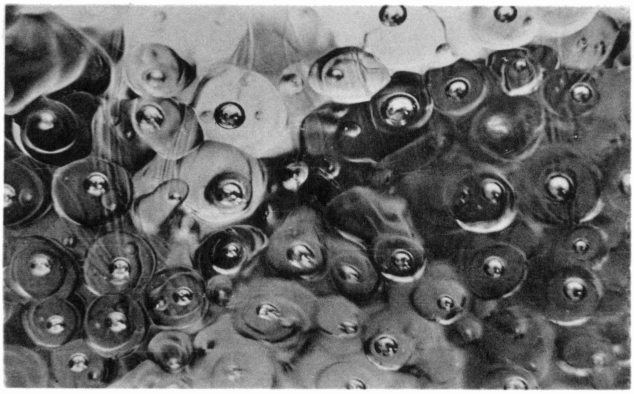


FIG. 2

FIG. 1. Appearance of the downward-facing side of a piece of ice, after attack by upward drilling gas bubbles, and after its removal from the alcohol solution.

FIG. 2. Downward view through ice of small bubbles drilling upwards on the lower ice surface in an alcohol solution. The edges of the pits shown in figure 2 are visible as approximately circular lines, and the gas bubble responsible for each hole can be seen in the interior of this circle.

FIG. 3. Rock salt crystal after attack by water, showing the downward facing surface. The formation of gas bubbles on the lower surface during attack results in a pitted condition of this surface, which is similar to that observed on ice and refractory blocks above.

3) The component of solution absorbed by the surface of the bubble, which is responsible for its lower surface tension, must be of some help in dissolving the solid; in the case of ice, it should form a freezing mixture with ice and water, i.e., it should lower the freezing point of the solution.

Upward Drilling by Small Bubbles.

The table shows specific gravity and surface tension for a number of substances. The importance of the above conditions are confirmed by the observations on these substances, which are given below.

	Specific Gravity	*Surface Tension*
Water	1.00	72 dyne/cm.
Methyl Alcohol	0.80	22.6
Acetic Acid	1.05	27.6
Ammonium Hydroxide (29° Be.)	0.88	56
Sodium Chloride (26% solution)	1.2	80
Sodium Oleate (0.1% solution)	1.0	30

a) *Alcohol and Water.* Mixtures of alcohol and water are lighter than water, their surface tension is lower, and they form freezing mixtures with ice. The conditions for upward drilling on ice are therefore satisfied. Small bubbles of gas, probably dissolved air, are released when alcohol is added to water, and, with concentrations of alcohol above five per cent these bubbles rapidly drill upwards on ice. Views of these small bubbles in their cavities in the lower surface are shown in Plate II, Fig. 2 and Text Fig. 1.

b) *Effect of Wetting Agents on Alcohol Solutions.* The action of small bubbles in alcohol solutions, which would otherwise drill rapidly on ice, is greatly retarded by the addition of wetting agents, such as soap or commercial detergents. Solutions of wetting agents by themselves will not drill upwards on ice. It is clear that, since the principal effect of these materials is on surface tension, they tend to prevent the absorption and transportation of alcohol by air bubbles. Very slight concentrations of wetting agents, of the order of 0.1 per cent, can decrease surface tension to 25 or 30 dyne/cm., which is the same as that of concentrated alcohol solutions. The amount of wetting agent carried through the region of dilute solution near the ice therefore cannot result in concentrations greater than about 0.1 per cent, which is not enough to affect seriously the solution of ice.

c) *Acetic Acid and Water.* The conditions for upward drilling are also met by this combination, except that its density is decreased by the addition of water. Acetic acid solutions, however, maintain upward drilling on ice for short periods of time. The conditions on the density of the solution is thus not so immediately necessary, although it is evident that, if it is not met, upward drilling cannot proceed very long.

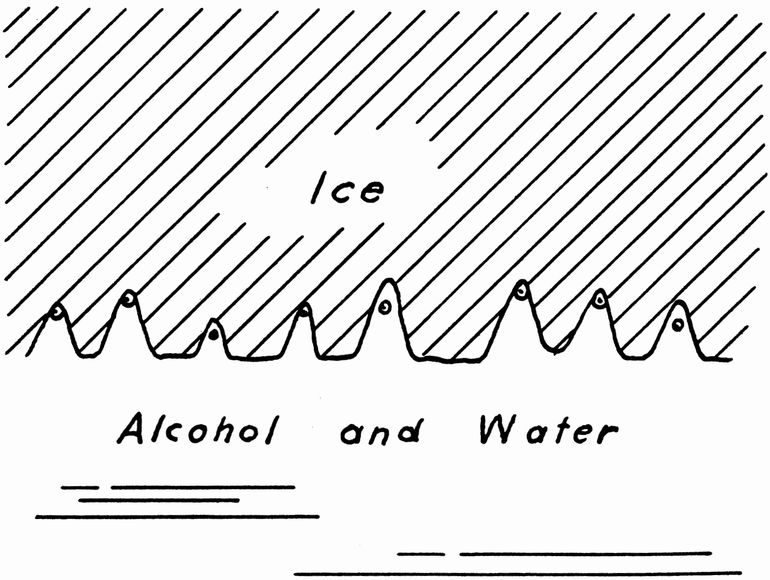


Fig. 1. Side view of small bubbles drilling upwards in ice, showing the development of pits caused by the attack of the bubbles.

d) *Ammonium Hydroxide.* Ammonia decreases the density and surface tension of water. The surface tension of 29° Be. ammonium hydroxide is about the same as that of a ten per cent alcohol solution, in which upward drilling can be observed. Bubbles introduced even into very concentrated ammonium hydroxide, however, will not initiate upward drilling on ice. In this case, the change in surface tension is so small, and the general attack by the concentrated solutions so rapid, that upward drilling does not occur.

e) *Salt Solutions.* Salt solutions are generally heavier than water and have higher surface tension. There is thus no reason to expect upward drilling on ice to occur, even in very concentrated solutions, and it has not been observed. However, in the attack by water on solid salts the above conditions are satisfied, and the surface of a bubble can absorb water and carry it to the solid. When rock salt is dissolving in water, attack is greatest at regions near air bubbles, and Plate II, Fig. 3 shows the appearance of a piece of rock salt after such an attack.

Upward Drilling by Large Bubbles In Alcohol Solutions.

We have so far discussed the behavior of small bubbles less than one mm. in diameter. Larger bubbles have a similar effect

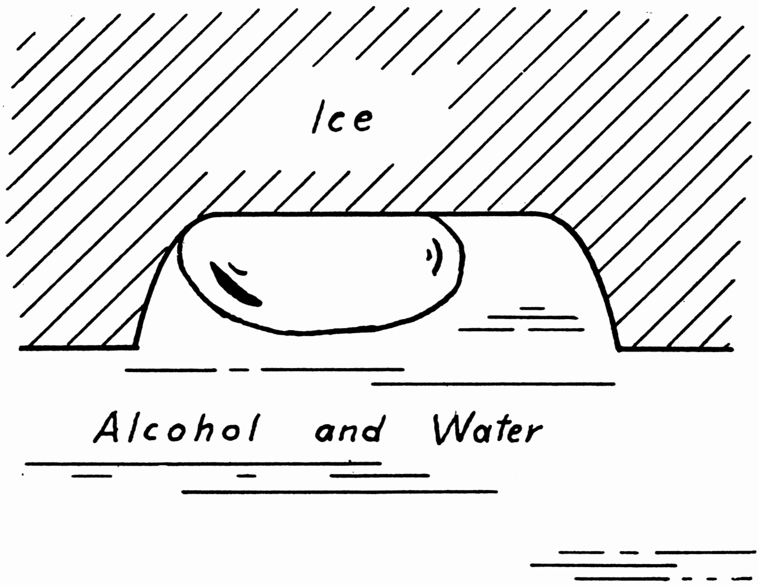


Fig. 2. Side view of a large bubble drilling upwards in ice. The top of the bubble flattens, and it drills by moving in a circle around the edge of the hole, always keeping in contact with the ice.

on solids, but they do not leave its lower surface. In drilling on ice, irregularities in the solid surface above the bubble are filled with liquid, so that the upper surface of the bubble is approximately flat. Attack by the bubble proceeds upwards and sideways, smoothing out the surface irregularities, and forming a cylindrical hole larger in diameter than the bubble, in which drilling is accomplished by rotation of the bubble about the center of the hole (Text Fig. 2). In alcohol solutions, attack by these larger bubbles is slower than by smaller ones, and bubbles of diameters of about one mm. are most effective in drilling.

As with smaller bubbles, which jumped downwards off the ice when absorbed alcohol was removed, the motion of the larger bubbles results from the tendency of the surface to lower its energy by absorbing alcohol but in this case, motion is horizontal instead of vertical. This motion is also partly due to the formation of liquid above the bubble, which accumulates there as the bubble attacks the ice. With very large bubbles drops of liquid may form above the bubble and eventually fall through it before it moves. These large bubbles, however, are unstable, showing a tendency to break up into smaller ones, particularly on a plane surface, and their rate of attack is very low.

Because of flattening of the bubble against the ice, the area of the capillary film between the ice and the bubble increases rapidly as the size of the bubble increases. This film will freeze if the ice is cold enough and the bubbles are large enough. At ice temperatures somewhat below freezing, for example, 20°F., even moderately sized bubbles freeze to the ice, and the general solution of the ice proceeds around it. Liquid pockets no longer form in cavities above the bubble, and, since there is no capillary film above the bubble, there is no transportation of alcohol to this area, and no upward drilling. The temperature of the ice, however, has no effect on the behavior of small bubbles, which drill upwards as fast on cold ice as on warm.

It should be noted that flux line attack, in which the solid dissolves rapidly at the meniscus of the free liquid surface on the solid, is to some extent another manifestation of the phenomenon observed in upward drilling. Ice floating in alcohol is subject to flux line attack, and it assumes the characteristic shape caused by this attack.[3] Solution proceeds rapidly at the meniscus, since alcohol is brought to this point by the variations in surface energy, and alcohol is removed continuously by melting ice at this point.

Motion of a Bubble, Taking Viscosity into Account.

As shown above the energy given to the bubble by surface tension forces is $\Delta\gamma\sigma$, or $4\pi R^2\Delta\gamma$, if the bubble is spherical with radius R. This appears as kinetic energy and, if the energy

[3] Preston, F. W.: Bull. Am. Cer. Soc. 15.409, 1936.

is given to the bubble instantaneously, it can be approximated by $\frac{1}{2}m^*v_0^2$, where m^* is $\frac{1}{2}$ the mass m of the fluid displaced by the bubble.[4] The initial velocity v_0 of the bubble is therefore given by

$$v_0^2 = 12\Delta\gamma/\rho R$$

where ρ is the density of the solution. We shall assume that the viscosity of the fluid can be represented by a resisting force proportional to velocity, and the equation of motion is then[5]

$$m^* v\, dv/dx = -mg - kv$$

where mg is the force of gravity and kv the retarding force due to the viscosity η of the liquid.

The constant k appearing in the equation of motion may be found from Stokes' Law, which gives for the steady velocity v produced by a force field g on a spherical bubble in a liquid of viscosity η

$$v = -2gR^2\rho/9\eta$$

The force on the bubble producing this velocity is mg, or $m(9\eta/2R^2\rho)v$, i.e., it is proportional to the velocity, and by comparison $k = m(9\eta/2R^2\rho)$.

The velocity of the bubble is therefore given in terms of its displacement, if $v = -v_0$ when $x = 0$, by

$$x = (v_0 - v)/2a - (g/2a^2)\log_e (av_0 + g)/(av + g)$$

where $a = (9\eta/2R^2\rho)$. The maximum displacement y obviously occurs when the velocity of the bubble is zero, and is found by setting $v = 0$ in the above equation; y is the vertical distance which the bubble travels through the liquid.

If the viscosity of the fluid is zero, the distance y traveled by the bubble becomes $v_0^2/4g$, which agrees with the value obtained from the energy relations. The effect of viscosity is

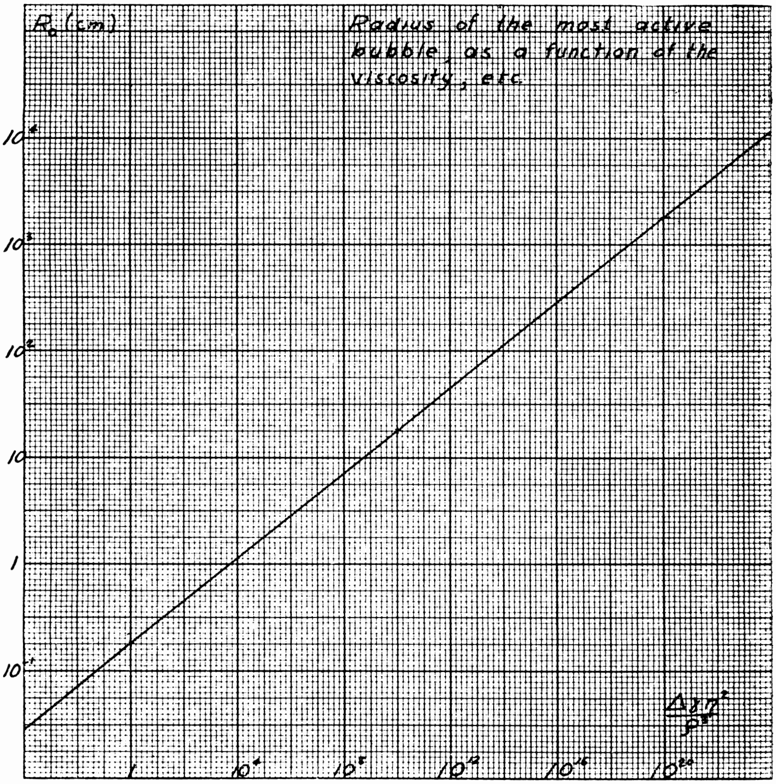
[4] The kinetic energy of translation of a sphere in an infinite ideal liquid, when the accompanying motion of the liquid is irrotational, is given by $T = \frac{1}{2}(\frac{1}{2}M' + M),\sigma^2$ where M is the mass of the sphere, M′ that of the displaced liquid (See, for example, Houston, R. A.: 'Introduction to Mathematical Physics', (Longmans, Green & Co., 1920) p. 56). Here we are dealing with a gas bubble, so that M is negligible compared to M′.

[5] For a bubble in an infinite ideal liquid, the Lagrangian $L = T - V$ is given by $L = \frac{1}{2}m^*v^2 - mgx$, where $V = mgx$ is the potential energy of the bubble in the gravitational field, and x measures distance downwards. The equation of motion $d/dt\,(\delta L/\delta\dot{x}) - \delta L/\delta x = O$ thus becomes

$$d/dt(m^*v) = -mg.$$

The dissipative force due to viscosity adds a term $-kv$ to the force of gravity $-mg$, and the independent variable x is used instead of t, according to the relation $\frac{d^2x}{dt^2} = v\frac{dv}{dx}$.

to decrease y from this maximum value, and, for finite viscosity y is zero for bubbles with radius either zero or infinity, in the first case because of the viscosity, and in the second because the surface tension forces become negligible compared with the mass of the bubble. For finite values of y therefore has a



maximum value y_o, which can be found by setting $dy/dR = 0$, and the corresponding value of R is the radius of the most active bubble, R_o.

We shall use the following substitutions:

$$\beta^2 = \frac{12\Delta\gamma}{\rho} \qquad a = 9\eta/2\rho \qquad c = a\beta/g$$

$$R = (c\xi)^{2/5}$$

so that

$$y = (c^{3/5}\beta/2a)\,[\xi^{3/5} - \xi^{8/5}\log_e(1+1/\xi)]$$

and

$$dy/dR = O = 8\xi^2(1+1/\xi)\log_e(1+1/\xi) - 8\xi - 3$$

The root of this equation can be found arithmetically to be $\xi_0 = 0.8685$, and hence $R_0 = (0.869\ c)^{2/5}$. The radius of the bubble having the maximum distance of travel in a liquid of viscosity η and density ρ with a change in surface tension $\Delta\gamma$, and the distance of travel y_0 are thus given by

$$\log_{10} R_0 = -0.744 + 1/5 \log_{10} (\Delta\gamma\eta^2/\rho^3)$$
$$\log_{10} y_0 = -2.005 + 1/5 \log_{10} (\Delta\gamma^2/\rho^2\eta^2)$$

The size of the most active bubble, R_0, is given in Text Fig. 3 as a function of surface energy change, viscosity, and density of the solution.

These equations can be applied to the three cases mentioned above; upward drilling in alcohol solutions on ice, which can be observed directly, attack on refractories by molten glass, and attack by viscous magmas on overlying rocks.

1) *Attack of Alcohol Solutions on Ice.*

The change in surface tension is of the order of 30 dyne/cm., the viscosity is 0.02 poises, and the density 1 gm/cc. Substituting above gives the values $R_0{}^{R_0} = 0.075$ cm., and $y_0 = 0.7$ cm. The most active bubble therefore should have a diameter of 1.5 mm., or about 1/16″, which agrees well with observation. The distance of travel is somewhat too large for bubbles of this diameter, although it is not off by more than a factor of 2.

This error might result from turbulence set up when the bubble starts to move. If velocity were high enough for this, we should have to take the resisting force proportional to v^2 instead of v. Reynold's number, using the maximum velocity v_0, is of the order of 300, which is well below the value (10,000) at which turbulence sets in, and we can probably neglect it. Reynold's number should be of the order of 1 or less, however, for Stokes' Law to hold accurately[6]; other errors are discussed below.

2) *Attack of Glass on Refractories.*

The change in surface tension produced by small additions of various substances to soda-lime-silica glass of average composition[7] has been measured by Badger, Parmelee and Williams[8] at 1250°C. According to these authors, the refractory mate-

[6] Lamb, H.: Hydrodynamics. Cambridge University Press, 1916, p. 589.
[7] SiO_2, 72.5%; Na_2O, 17.4%; CaO, 10.1%.
[8] Badger, Parmelee and Williams: Jr. Am. Cer. Soc. 20, 325, 1937.

rials,[9] alumina and silica, increase surface tension, and the fluxes, which are responsible for attack, lower surface tension. Thus one per cent addition of alumina increases surface tension 2.5 dyne/cm., and one per cent addition of soda decreases it 1.5 dyne/cm. These changes are in the right direction to produce upward drilling, since there is a possibility of absorption and transportation of flux by the surface of the bubble.

The difference in surface tension between glass very near the refractory surface and that further away from it may be of the order of 20 dyne/cm. The density is 2 gm/cc., and the viscosity about 100 poises. These figures give for R_o, 1.4 cm., and for y_o, 0.10 cm.

The calculated size of the most active bubble is thus about the same (one inch) as the size of the pits found in refractory blocks taken from a glass tank. Upward drilling undoubtedly is the cause of these pits found in the downward facing surface. Because of the effect of fluxes and refractory materials on the surface tension of glass, however, it does not seem that very much can be done to diminish this type of attack. Possibly some material, which is not itself a flux but which reduces surface tension, can be found and added either to the refractory block or to the glass melt. This material would tend to be absorbed on free surfaces of glass, and would keep the fluxes away from the refractory, just as wetting agents, for example, tend to keep alcohol away from ice.

3) *Attack of Viscous Magmas on Overlying Rocks.*

The best values of surface tension change and viscosity are rather uncertain in this case. R_o has therefore been computed for the following values of η and $\Delta\gamma$.

Radius of the Most Active Bubble (meters).

		Viscosity (Poises)			
		10^5	10^7	10^9	10^{11}
Change in	1.0 dyne/cm.	0.12	0.75	4.8	30
Surface Tension	10 "	0.19	1.2	7.5	48

[9] Glass tank refractory blocks consist mainly of silica and alumina; the ratio of alumina to silica varies from ¼ to ¾, and new refractories of even higher ratios have recently been introduced. The low ratio refractories consist of meta-kaolin fired to high temperatures; the high ratio are a mixture of mullite and carborundum crystals, bonded by a high silica glass.

High values of viscosity, which can be reasonably assumed for magmas at depth, evidently lead to maximum activity by bubbles which are of the order of the bore of volcanoes. We must, however, leave to the geologist the discussion of the possibility of occurrences of this kind of attack by viscous magmas.

Conclusion.

Computations of the radius of the bubble of maximum activity based on the above equations are of the right order of magnitude, not only in the case of upward drilling by bubbles on ice, but also in the case of refractory attack by glass and of the attack of magmas on rock. They are best in the case of upward drilling on ice, however; they are less certain for refractory attack, and still less for attack by magmas, both because the correct values of the physical constants involved in the latter cases are less well known, and because the motion of the bubble under our assumptions is only an approximation to the real motion.

The equation of motion of the bubble is based on the assumption of a constant 'mass' for the bubble. This correctly describes the motion of a bubble suspended in an infinite liquid, and, for viscous liquids and small velocities, it leads to Stokes' Law giving the relation between acceleration and velocity. The presence of boundaries, however, has a large effect on the motion of the bubble, which is accompanied by motion of liquid a great distance from it. In this case, the kinetic energy of the liquid is not given by the energy of a bubble with a fictitious mass of half the displaced liquid. For the same reason, Stokes' Law is not valid in the presence of boundaries near the bubble.

The formulae computed are to a great extent extrapolations, based on these somewhat questionable approximations. In spite of this, it is felt that upward drilling is correctly described by motion of the bubble, having its origin in variations in surface energy, which is damped due to viscosity. Because of this damping, we may expect a certain size of bubble to be the most active in drilling. The computation thus is somewhat lacking in accuracy with regard to detailed application, but, in a general way, it corresponds to the physics of the problem, and the results seem to be in the right order of magnitude.

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