ART. LII. — *On the Relative Accuracy of Different Methods of Determining the Solar Parallax*; by WM. HARKNESS.

[The substance of this paper was read before the American Association for the Advancement of Science, at Cincinnati, August, 1881.]

THE object of this paper is to compare the various methods of determining the solar parallax, and to show that the photographic method employed by the United States Transit of Venus Parties in 1874 is among the most accurate known, and should not be neglected in observing the transit of 1882.

The following notation will be employed in algebraic formulæ:

a =mean distance of the earth from the sun.
a_1 =that distance between the earth and the sun which would satisfy Kepler's third law.
a_2 =mean distance of the earth from the moon.
c =a constant such that $c\rho=\rho_1$.
E =the mass of the earth.
e =eccentricity of the moon's orbit.
e_1 =eccentricity of the earth's orbit.
G =observed force of gravity at a point upon the surface of the earth.
k =Gauss's constant for the solar system.
L =constant of the earth's lunar inequality.
l =length of simple pendulum.
M =the mass of the moon.
m =ratio of the mean motions of the sun and moon =0·07480133.
P =the constant of lunar parallax =3422″·7.
P_1=that value of the constant of lunar parallax which would satisfy Kepler's third law.
p =the constant of solar parallax.
Q =the parallactic inequality of the moon.
S =the mass of the sun.
s =geocentric latitude of the moon.
T =length of the sidereal year, expressed in seconds of mean time =31,558,149^s.
T_1=length of the sidereal month, expressed in seconds of mean time =2,360,591^s·8.
t =time.
V =the velocity of light.
α =the constant of aberration.
γ =Delaunay's constant, which is approximately sin ½ (inclination of lunar orbit to plane of ecliptique), and the exact value of which is 0·04488663. See DTL., vol. ii, 802.
θ =the time taken by light to traverse the mean radius of the earth's orbit.
μ =motion of moon's node, relatively to the line of equinoxes, in 365¼ days.
ν =the heliocentric longitude of the earth.

ν' = the geocentric longitude of the moon.
ρ = the equatorial radius of the earth.
ρ_1 = radius of the earth at latitude φ.
φ = geocentric latitude.
Ψ = the luni-solar precession.
Ω = the constant of nutation.

In citing authorities the following abbreviations will be used:

MAc = Memoires de l'Académie Royale des Sciences. Paris.
HAc = Histoire de l'Académie Royale des Sciences. Paris.
CRH = Comptes Rendus Hebdomadaires des séances de l'Académie des Sciences. Paris.
PTr = Philosophical Transactions of the Royal Society of London.
ANn = Astronomische Nachrichten.
MAS = Memoires of the Royal Astronomical Society. London.
MNt = Monthly Notices of the Royal Astronomical Society, London.
OPM = Annales de l'Observatoire Impérial de Paris. Mémoires.
WOb = Astronomical and Meteorological Observations made at the United States Naval Observatory. Washington.
PTL = Théorie du Mouvement de la Lune, par Jean Plana. Turin, 1832. 3 vols. 4to.
DTL = Théorie du Mouvement de la Lune, par Ch. Delaunay. Paris, 1860–1867. 2 vols. 4to.

Every known method of determining the solar parallax belongs to one or other of the following classes, namely:

I. Trigonometrical methods.
II. Gravitational methods.
III. Photo-tachymetrical methods.

We will consider them in their order.

Trigonometrical Methods.

Observations of Mars, when in opposition to the sun, and at its least distance from the earth, constitute one of the oldest trigonometrical methods of determining the solar parallax. There are two ways of making the observations. Either the planet is observed on or near the meridian, at two stations, situated respectively in the northern and southern hemispheres; or it is observed soon after rising, and just before setting, at a single station. The first method will be termed the meridian method; the second, the diurnal method. In the meridian method the observations may be made either with a transit circle, or with a micrometer attached to an equatorial telescope. In the diurnal method they may be made either with an equatorial telescope, or with a heliometer.

The values of the solar parallax resulting from some of the most noteworthy attempts by the meridian method are as follows:

1672. J. D. Cassini (MAc, viii, 114), ... 9″·5
1751. Lacaille (Ephémérides des Mouvements Célestes depuis 1765 jusqu'en 1774. Paris. Introd. p. 1), 10·38
1835. Henderson (MAS, viii, 103), ... 9·028
1856. Gilliss and Gould (U. S. Ast. Ex. to the South. Hemisphere, vol. iii, p. cclxxxviii), ... 8·495
1863. Winnecke (ANn, Bd. lix, s. 264), ... 8·964
1865. E. J. Stone (MAS, vol. xxxiii, p. 97), ... 8·943
1865. A. Hall (WOb, 1863, App. p. lxiv), ... 8·842
1867. Newcomb (WOb, 1865, App. II, p. 22), ... 8·855
1879. Downing (ANn, Bd. xcvi, s. 127), ... 8·960

The following are some of the results from the diurnal method:

1672. J. D. Cassini (MAc, viii, 107), ... 10″·2
1672. Flamstead (PTr, 1672, p. 5118), ... 10.
1719. Bradley and Pound (Gehler's Physikalisches Wörterbuch, viii, 822), ... 10·5
1857. W. C. Bond (Gould, Ast. Jour., v, 53), ... 8·605
1877. Maxwell Hall (MAS, vol. xliv, p. 121), ... 8·789
1879. Gill (MNt, 1879, vol. xxxix, p. 437), ... 8·78

Owing to the comparative nearness of the asteroids, and their small, well defined disks, it has been thought that the solar parallax might be accurately derived from observations made upon them in the manner just described for Mars. So far as I know, the following are the only attempts which have been made in that direction:

1875. Galle, from Flora (ANn, Bd. lxxxv, s. 267), ... 8″·879
1877. Lindsay and Gill, from Juno (Dunecht Observatory Publications, vol. ii, 211), ... 8·765

The same method has also been applied to Mercury and Venus, but there are great difficulties in the way of obtaining satisfactory results from these planets.

Transits of Venus.—Until quite recently, astronomers have believed that transits of Venus furnish by far the most accurate means of determining the solar parallax. Such transits have been observed by three different methods, namely: 1. By noting the times of contact between the limbs of Venus and the sun. 2. By observing the position of Venus upon the sun's disk with a heliometer. 3. By photographing the sun with Venus upon its disk, and subsequently measuring the photographs.

Contact observations.—The following are some of the results for solar parallax obtained by different astronomers from contact observations of the transits of Venus in 1761, 1769 and 1874:

TRANSIT OF 1761.

1763.	Hornsby (PTr, 1763, p. 494),	9″·73
1763.	Short (PTr, 1763, p. 340),	8·56
1765.	Pingré (HAc, 1765, p. 32),	10·10
1767.	Planman (PTr, 1768, p. 127),	8·49

TRANSIT OF 1769.

1770.	Euler (Novi Commentarii Ac. Sc. Petropol., t. xiv),	8″·8
1771.	Hornsby (PTr, 1771, p. 579),	8·78
1771.	Lalande (HAc, 1771, p. 798),	8·62
1771.	Maskelyne,	8·723
1772.	Lexell,	8·63
1772.	Pingré (HAc, 1772, p. 419),	8·80
1772.	Planman,	8·43
1814.	Delambre (Astron. Théorique et Pratique, t. i, p. xliv),	8·552
——.	Du Sejour (Traité Analytique des Mouvements Apparent des Corps Celestes, t. i, pp. 451–491),	8·85
1832.	Ferrer (MAS, v, 286),	8·58
1865.	Powalky (Conn. de Temps 1867 Additions, p. 22),	8·832
1868.	E. J. Stone (MNt, vol. xxviii, p. 264),	8·91

TRANSITS OF 1761 AND 1769.

1835.	Encke (Abhand. der Akad. zu Berlin, 1835, Math. Kl., s. 309),	8·571

TRANSIT OF 1874.

1877.	Airy (The Observatory, 1877, vol. i, p. 149),	8·760
1878.	Tupman (MNt, 1878, vol. xxxviii, p. 455),	8·846

The large differences in the parallaxes obtained by different astronomers from the same observations are due to the circumstance that, as the instants of contact are rendered uncertain by the intervention of various disturbing phenomena, many of the observers record two or three different times, corresponding to as many different phases which they endeavor to describe, and thus the resulting parallaxes are influenced to a certain extent by the interpretation put upon these descriptions. The interior contacts give better results than the exterior ones, but in any case the probable error is large. From sixty-one selected observations of interior contacts of the transit of December, 1874, discussed by Col. Tupman (MNt, 1878, vol. xxxviii, 20 on page 450, and 41 on p. 453), I find the probable error of an observed time of contact to be $\pm 4^s{\cdot}59$, which corresponds to a probable error of $\pm 0''{\cdot}15$ in the distance between the centers of the sun and Venus. Actual errors of from twenty to thirty seconds in the observed times of contacts are by no means uncommon.

Observations with heliometers.—A few heliometers were used in observing the transit of December, 1874, but I am not aware that anything has yet been published which suffices to show how accurately they will furnish the solar parallax.

Photographic observations.—For observing the last transit of Venus there were used at least two kinds of photoheliographs, constructed upon widely different principles. In what follows I shall consider only the results yielded by apparatus of the kind used by the United States Transit of Venus parties.

As the reductions of the United States transit of Venus observations are not yet quite completed, it is impossible to say exactly what degree of accuracy the photographs will give; but fortunately the same instruments which were used in December, 1874, to observe the transit of Venus at Kerguelen Island, Hobart Town and Peking, were used in May, 1878, to observe the transit of Mercury at Cambridge, Mass., Washington, D. C. and Ann Arbor, Mich.; and as the transit of Mercury photographs are completely reduced, Rear Admiral John Rodgers, Superintendent of the Naval Observatory, has kindly authorized me to make use of the results. They are as follows:

The total number of plates measured was 119, of which 25 were made at Cambridge, 30 at Washington, and 64 at Ann Arbor. Each plate was measured by two different persons. The errors to be considered are of four different kinds, namely: constant and accidental errors in measuring the plates, and constant and accidental errors peculiar to each station.

Each plate having been measured in duplicate, if the positions of Mercury upon the sun's disk given by the measures of the first observer are subtracted from those given by the measures of the second observer, the mean of all the residuals thus obtained will be the constant error due to personal equation in reading. Its amount for each station is

	In altitude.	In azimuth.
Cambridge	$-0''\cdot10$	$-0''\cdot08$
Washington	$-0\cdot09$	$+0\cdot08$
Ann Arbor	$+0\cdot15$	$-0\cdot02$

Thus it appears that, for the mean of the three stations, the constant error of reading is practically zero.

If the mean of the readings by the two observers is accepted as the truth, the probable error of the position of Mercury upon the sun's disk, as determined from a single set of readings by one observer, is

	In altitude.	In azimuth.
Cambridge	$\pm0''\cdot18$	$\pm0''\cdot20$
Washington	$\pm0\cdot19$	$\pm0\cdot18$
Ann Arbor	$\pm0\cdot24$	$\pm0\cdot28$

The locus of the average probable error of reading therefore lies within a circle whose radius is $0''\cdot21$.

The corrections found at each station to LeVerrier's tables of Mercury, as represented by the British Nautical Almanac for 1878, are

	R. A.	N. P. D.
Cambridge............	$+0^s{\cdot}079$	$-0''{\cdot}22$
Washington...........	$+0{\cdot}105$	$-0{\cdot}12$
Ann Arbor............	$+0{\cdot}083$	$+0{\cdot}47$

The correction to the north polar distance, given by the Ann Arbor plates, seems to be affected by a systematic error, but it is doubtful if its source can be discovered because no details of the observations were sent to the Naval Observatory, and Professor Watson, who made them, is now dead.

The probable error of a position of Mercury depending upon two sets of readings made upon a single photograph is

	R. A.	N. P. D.
Cambridge............	$\pm 0''{\cdot}570$	$\pm 0''{\cdot}562$
Washington...........	$\pm 0{\cdot}655$	$\pm 0{\cdot}579$
Ann Arbor............	$\pm 0{\cdot}436$	$\pm 0{\cdot}514$

The probable errors in right ascension having been reduced to arc of a great circle. We may infer from the mean of all the stations that the average locus of the probable error of the position of the planet in the heavens is a circle whose radius is $0''{\cdot}553$.

To exhibit yet more clearly the degree of accuracy attained by the photographic method, a table is appended, which includes all the plates, and shows the number of residuals, both in right ascension and north polar distance, which fall between $0''{\cdot}0$ and just under $0''{\cdot}2$, $0''{\cdot}2$ and just under $0''{\cdot}5$, etc. In tabulating the right ascension residuals it has been assumed that $0''{\cdot}2 = 0^s{\cdot}01$, $0''{\cdot}5 = 0^s{\cdot}03$, $1''{\cdot}0 = 0^s{\cdot}07$, $1''{\cdot}5 = 0^s{\cdot}10$, $2''{\cdot}0 = 0^s{\cdot}13$.

Limits.	Cambridge.		Washington.		Ann Arbor.	
	R. A.	N. P. D.	R. A.	N. P. D.	R. A.	N. D. P.
$0''{\cdot}0$–$0''{\cdot}2$	3	5	3	7	11	11
0·2–0·5	5	6	5	6	16	14
0·5–1·0	10	7	11	10	29	27
1·0–1·5	5	4	8	3	5	7
1·5–2·0	0	2	2	1	3	5
2·0 and over	2	1	1	3	0	0

Theory of the Gravitational Methods.

We begin the consideration of the gravitational methods by deriving an expression for the solar parallax in terms of the earth's mass.

If l is the length of a simple pendulum which makes one vibration in t seconds of mean time, the observed force of gravity will be

$$G = \frac{\pi^2 l}{t^2} \qquad (1)$$

The attraction of the earth at a point upon its surface in geocentric latitude φ is

$$\frac{k^2 E}{\rho_1^{\,2}} \qquad (2)$$

The observed force of gravity is the earth's attractive force diminished by the resolved value of its centrifugal force. At the equator the centrifugal force is $G \div 289{\cdot}24$, while in any other latitude it is $G \cos\varphi \div 289{\cdot}24$; and the resolved part of this force acting in the direction of the vertical is $G \cos^2\varphi \div 289{\cdot}24$. Equating the earth's attraction to the force of gravity augmented by the centrifugal force, we have

$$\frac{k^2 E}{\rho_1^{\,2}} = G\left(1 + \frac{\cos^2\varphi}{289{\cdot}24}\right) \qquad (3)$$

Whence, by (1)

$$\frac{k^2}{\pi^2} = \frac{\rho_1^{\,2} l}{t^2 E}\left(1 + \frac{\cos^2\varphi}{289{\cdot}24}\right) \qquad (4)$$

If T is the length of the sidereal year, expressed in seconds of mean time, and a_1 is that value of the semi-major axis of the earth's orbit which would satisfy Kepler's third law, we have

$$T^2 = \frac{4\pi^2 a_1^{\,3}}{k^2(S+E)} \qquad (5)$$

Le Verrier has shown that $a = 1{\cdot}000141 a_1$, (OPM, ii, 60, and iv, 103). Substituting this value in (5), and transposing

$$\frac{k^2}{\pi^2} = \frac{4a^3}{T^2(S+E)\,(1{\cdot}000141)^3} \qquad (6)$$

Eliminating k and π between (4) and (6), and rearranging the terms

$$\frac{S+E}{E} = \frac{4t^2 a^3}{l\,T^2 \rho_1^{\,2} (1{\cdot}000141)^3 \left(1 + \dfrac{\cos^2\varphi}{289{\cdot}24}\right)} \qquad (7)$$

Owing to the equatorial bulging of the earth, the points which have $\sqrt{\frac{1}{3}}$ for the sine of their geocentric latitude are the only ones upon the surface of the earth at which a pendulum will vibrate as it would if the whole mass of the earth were concentrated at its center. For that reason we take $\sin^2\varphi = \frac{1}{3}$, and consequently $\cos^2\varphi = \frac{2}{3}$. We also put $\rho_1 = c\rho$, and $a \sin p = \rho$. Substituting these values in (7), it becomes

$$\frac{S+E}{E} = \frac{4t^2\varphi}{lT^2c^2\sin^3 p(1\cdot000141)^3\left(\frac{434\cdot86}{433\cdot86}\right)} \tag{8}$$

The equation $\sin^2\varphi = \frac{1}{3}$, gives $\varphi = 35° 15' 52''$. Adding to this the angle of the vertical, 10′ 51″, the geographical latitude is 35° 26′ 43″, and the corresponding value of log c is 9·999515. If we take $t = 1^s$, the value of l for latitude 35° 26′ 43″ is 0·992732 meters.* Substituting these values, together with T=31,558,149 seconds of mean solar time, and ρ=6,378,390 meters, in equation (8), it becomes

$$p^3\left(\frac{S+E}{E}\right) = 226{,}350{,}000 \tag{9}$$

or

$$p = 609\cdot434\sqrt[3]{\frac{E}{S+E}} \tag{10}$$

where p is expressed in seconds of arc.

In connection with equations (9) and (10) the reader may compare "Hansen on the calculation of the sun's parallax from the lunar theory," MNt, 1864, vol. xxiv, p. 11; "Darlegung der theoretischen Berechnung der in den Mondtafeln angewandten Störungen, von P. A. Hansen." Zweite Abhandlung, s. 271; "E. J. Stone on the value of the solar parallax, as deduced from the parallactic inequality in the earth's motion." MNt, 1868, vol. xxviii, p. 23; Le Verrier, in the CRH, 1872, t. lxxv, p. 166, and MNt, 1872, vol. xxxii, p. 322.

The equation of the parallactic inequality of the moon's motion, as given by Newcomb from the theories of Plana and Delaunay, is

$$Q = 0\cdot24123\frac{1-M}{1+M}\times\frac{p}{\sin P(1-\frac{1}{6}m^2)} \tag{11}$$

Substituting the numerical values of P and m, and transposing, this becomes

$$p = [8\cdot837088]\,Q\frac{1+M}{1-M} \tag{12}$$

from which p can be found when Q and M are known. The quantity within the square brackets is the logarithm of the number which it represents.

In connection with equations (11) and (12) the reader may compare PTL, t. iii, p. 13; DTL, t. ii, p. 847, equation 342; WOb, 1865, Appendix 2, p. 24; MNt, 1880, vol. xl, p. 468.

The lunar equation of the earth's motion is (OPM, iv, 47)

$$\delta\nu = -\frac{M}{E+M}\times\frac{\sin p'}{\sin P'}\times\cos s'\sin(\nu'-\nu) \tag{13}$$

* Everett, Units and Physical Constants, p. 21.

in which p' and P' are the actual values of the solar and lunar parallaxes at the instant for which $\delta\nu$ is required. For any given lunation, $\delta\nu$ will evidently attain its maximum value when $\sin(\nu'-\nu)=1$, that is, when the longitudes of the sun and moon differ by ninety degrees. If now we have an extensive series of observed values of $\delta\nu$, covering many complete revolutions of the moon's node; $\delta\nu$ will have assumed all possible values, the mean of which will be the constant of the lunar inequality; p' will have assumed all possible values, the mean of which will be the constant of solar parallax; and the moon will have had all possible latitudes, the mean of which will be zero. With P' the case will be somewhat different. It is equal to the constant of lunar parallax, plus a series of terms multiplied by factors made up of the mean anomaly of the sun, the mean anomaly of the moon, the mean distance of the moon from its ascending node, and the difference of the mean longitudes of the sun and moon. All these terms, except those involving the difference of the mean longitudes, will assume all possible values and vanish from the mean. The mean of all the values of P' will therefore be, P + terms depending upon the difference of mean longitudes of the sun and moon.* Turning now to the second volume of Delaunay's theory of the moon, we find that the only term of this kind in the lunar parallax is the one numbered (27), upon page 917, and its value is $28''\cdot1788 \cos 2D$. As we have supposed all our observations of $\delta\nu$ to be made when D was 90°, the value of this term will be $-28''\cdot18$, and the mean value of P' will be $P-28''\cdot18 = 3394''\cdot52$. Substituting the mean values thus found in (13), and rearranging the terms, we obtain

$$p = 0\cdot0164564\,L\left(\frac{E+M}{M}\right) \qquad (14)$$

In connection with equation (14) the reader may compare, Le Verrier, OPM, iv, 100; Newcomb, WOb, 1855, App. II p. 28; E. J. Stone, MNt, 1868, vol. xxviii, p. 24.

The Moon's Mass.

Before the solar parallax can be obtained from equations (12) and (14), it is necessary to know the moon's mass. Let us consider the different ways of determining it.

The first determination of the moon's mass was made from the tides, by Newton, in 1687. Since then other investigators have employed the same method, but owing to the theoretical and practical difficulties inherent in it, their results have been so discordant as to command very little confidence. Perhaps

* In strictness it should be the difference of the *true* longitudes of the sun and moon.

the most trustworthy result is that by Mr. Wm. Ferrel of the United States Coast Survey, who found the moon's mass from the tides at Brest $\frac{1}{77.14}$, and from the tides at Boston $\frac{1}{78.64}$, the most probable mean being $\frac{1}{77.5}$. (Jour. Frank. Inst., 1871, vol. lxi, p. 366.)

In 1755, D'Alembert determined the moon's mass from the phenomena of precession and nutation, but to do this with extreme accuracy seems a difficult matter. The most recent attempt is by Mr. E. J. Stone (MNt, 1868, vol. xxviii, p. 43), who considers that his equations are accurate to terms of the third order in the lunar theory. With some changes of notation, they are

$$\left.\begin{aligned} \varepsilon &= \frac{Ma^3}{Sa_2^{\ 3}} \\ \Psi &= A\varkappa + B\varkappa\varepsilon \\ \Omega &= C\varkappa\varepsilon \end{aligned}\right\} \qquad (15)$$

in which

$$\left.\begin{aligned} A &= 1 + \frac{3e_1^{\ 2}}{2} \\ B &= 1 + \frac{3e^2}{2} - 6\gamma^2 \\ C &= \frac{2\gamma}{\mu}\left(1 + \frac{e^2 3}{2} - \frac{5\gamma^2}{2}\right) \end{aligned}\right\} \qquad (16)$$

Elimiting $\varkappa$ and ε from the equations (15), and introducing the sines of the parallaxes instead of the mean distances, we get

$$M = \frac{\sin^3 p . A\Omega S}{\sin^3 P \,(C\,\Psi - B\Omega)} \qquad (17)$$

which becomes

$$M = \frac{[2{\cdot}411505]\, A\Omega}{\sin^3 P\,(C\,\Psi - B\Omega)} \qquad (18)$$

by substituting the value of S $\sin^3 p$ from (9). The number within the square brackets is the logarithm of the quantity which it represents. Ten must be subtracted from its characteristic.

We will take

$\gamma = 0{\cdot}04488663$
$e = 0{\cdot}0548993$
$e_1 = 0{\cdot}0167711$
$\mu = -19^\circ\, 21'\, 20'' = -0{\cdot}337818$ of radius.
$P = 3422''{\cdot}7$

The value here given for e is that used by Delaunay (DTL, ii, 802). The value of P is that found from the Greenwich and Cape of Good Hope observations by Breen (MAS, 1864, vol. xxxii, p. 137) and E. J. Stone (MAS, 1866, vol. xxxiv, p. 16). Substituting these values in (16) and (18), the latter equation becomes

$$\frac{1}{M} = 47{\cdot}0243 \frac{\Psi}{\Omega} - 175{\cdot}705 \tag{19}$$

In connection with equations (18) and (19), the reader may compare PTL, t. iii, pp. 25–29; LeVerrier, OPM, t. iv, p. 101; Serret, OPM, v, 324; Newcomb, WOb, 1865, App. II, p. 28.

About 1795 Delambre seems to have determined the moon's mass from the lunar inequality of the earth's motion. This involves the use of equation (14), but as we propose to employ that equation for determining the solar parallax, we cannot avail ourselves of it for the mass of the moon.

There is yet another way of determining the moon's mass; to wit, by comparing the fall of heavy bodies at the surface of the earth with the fall of the moon in its orbit. The resulting equation will be similar to (8), except that for the masses of the sun and earth we must substitute the masses of the earth and moon, and instead of 1·000141 sin p we must employ the particular value of P which satisfies equation (5) when E+M is substituted in it for S+E, and T is taken to be the length of a sidereal revolution of the moon, expressed in seconds of mean time. Designating these special values of T and P by T_1 and P_1, we have

$$\frac{E+M}{M} = \frac{4t^2\rho}{lT_1^{\,2}\, c^2 \sin^3 P_1\left(\frac{434{\cdot}86}{433{\cdot}86}\right)} \tag{20}$$

Of the four methods just described for determining the moon's mass, that depending upon the tides is not sufficiently accurate, and that depending upon the lunar inequality of the earth's motion is not available, for our purpose. There remain only the two methods represented respectively by equations (19) and (20). Let us see what results they give.

As the luni-solar precession increases continually with the time, its value is now known very accurately. I adopt for it the numbers used by Messrs. Newcomb and Stone (WOb, 1865, App. II, p. 28; MNt, 1868, vol. xxviii, p. 43), namely 50″·378. The constant of nutation is much more uncertain. The following are some of the best modern values:

1842. C. A. F. Peters (Num. Con. Nut., p. 37), ... 9″·223
1844. C. A. F. Peters (Mem. Ac. Sc. St. Petersbourg, 7e sér. t. iii, p. 125), ... 9·216

1856.	LeVerrier (OPM, t. ii, p. 174),	9·23
1869.	E. J. Stone (MAS. vol. xxxvii, p. 249),	9·134
1872.	Nyrén (Mem. Ac. Sc. St. Petersbourg, 7e sér. t. xix, No. 2),	9·236

With $\Psi = 50''\cdot378$, formula (19) gives the mass of the moon corresponding to three different values of the nutation constant as follows:

$$\Omega = 9''\cdot230 \quad M = \frac{1}{80\cdot96}$$

$$\Omega = 9''\cdot223 \quad M = \frac{1}{81\cdot15}$$

$$\Omega = 9''\cdot134 \quad M = \frac{1}{83\cdot65}$$

The change in the moon's mass produced by a small change in the constant of nutation is given by the expression

$$d\left(\frac{1}{M}\right) = -28\cdot1\, d\Omega \qquad (21)$$

In view of the fact that Peters attributed a probable error of $\pm 0''\cdot0154$ to his most careful determination of the nutation constant, and in view of the subsequent widely differing determination by E. J. Stone, it can scarcely be supposed that the true value of the nutation is known within $\pm 0''\cdot02$. This corresponds to an uncertainty of $\pm 0\cdot56$ in the reciprocal of the moon's mass.

The length of the sidereal month is 2,360,591·8 seconds of mean solar time. Assuming the observed value of the constant of lunar parallax to be $3422''\cdot7$, Plana's theory gives $3419''\cdot62$, and Delaunay's theory $3419''\cdot59$, for the value of P_1. I adopt $3419''\cdot6$. Substituting these values in formula (20), the resulting mass of the moon is $\frac{1}{81\cdot77}$, and the change in the mass produced by a small change in the adopted parallax is given by the expression

$$d\left(\frac{1}{M}\right) = 5\cdot925\, dP \qquad (22)$$

The value of the lunar parallax now generally adopted, depends upon the investigations of Messrs. Breen and E. J. Stone. The results of these two gentlemen agree within $0''\cdot01$. The probable error of Mr. Breen's result is not stated, while that of Mr. Stone's is $\pm 0''\cdot049$. Nevertheless, it is not unlikely that the parallax may be one or two-tenths of a second in error. An error of $0''\cdot1$ would produce an error of 0·59 in the reciprocal of the mass.

Probably the moon's mass is about $\frac{1}{81\cdot5}$, but it is quite possible that this estimate may be in error by one part in a hundred. The precession-nutation method is considered one of the best for obtaining the moon's mass, but equations (21) and (22) show that neither it, nor the method by the fall of the moon in its orbit, is likely ever to furnish the mass within one part in a thousand. Throughout all his lunar work Hansen adopted a mass of $\frac{1}{80}$, and in what follows I will assume that the true mass lies between the limits $\frac{1}{80}$ and $\frac{1}{83}$.

Parallax from Gravitational Methods.

Mass of the Earth.—In 1872 LeVerrier obtained the mass of the earth from the inequalities in the motions of Venus and Mars, and the secular variations in the elements of their orbits, produced by it; and from the mass thus found he derived the solar parallax by means of an equation similar to (10). (CRH, 1872, t. lxxv, pp. 165–172; MNt, 1872, vol. xxxii, pp. 322–328.) He gave the resulting parallaxes without directly stating the masses, but it is readily seen that his values were as follows:

(A). From the latitudes of Venus at the moments of the transits in 1761 and 1769, earth's mass $= \frac{1}{325{,}165}$.

(B). From a discussion of the meridian observations of Venus in an interval of one hundred and six years, earth's mass $= \frac{1}{324{,}575}$.

(C). From observations of the occultation of ψ^2 Aquarii by Mars, October 1st, 1672, earth's mass $= \frac{1}{323{,}746}$.

Substituting these values in equation (10), the resulting values of the solar parallax are

A.	8″·862
B.	8·868
C.	8·875

Taking the earth's mass as unity, the change in the parallax produced by a change of one thousand units in the mass of the sun is given by the expression

$$dp = 0\cdot00912\, dS \qquad (23)$$

It is difficult to estimate the probable error of the above values of the earth's mass, but Tisserand seems to think it may be sufficient to affect the parallax by $\pm 0''\cdot07$. (CRH, 1881, t. xcii, p. 658.) As the secular variations of the elements of the orbits of Venus and Mars increase continually, they will ultimately attain sufficient magnitude to give a very exact value of the earth's mass, and then this method will furnish the solar parallax with the utmost precision.

Parallactic Inequality.—Professor Newcomb found that the value of the parallactic inequality of the moon deduced by Hansen from the Greenwich and Dorpat observations is $126''\cdot46$. (WOb, 1865, App. II, p. 23.)

From 2075 Greenwich lunar observations, made between 1848 and 1866, Mr. E. J. Stone found the parallactic inequality to be $125''\cdot36 \pm 0''\cdot4$; the probable error being estimated. (MNt, 1867, vol. xxvii, p. 271.)

From the Washington lunar observations, made between 1862 and 1865, Professor Newcomb found the parallactic inequality to be $125''\cdot46$. (WOb, 1865, App. II, p. 24.)

From an extended discussion of the whole subject, published in the MNt, 1880, vol. xl, pp. 386 to 411, and 441 to 472, Messrs. Campbell and Neison found the observed value of the parallactic inequality to be (p. 467) either $125''\cdot64 \pm 0''\cdot09$, or $124''\cdot64 \pm 0''\cdot25$; the difference arising from the admission or non-admission into the lunar theory of a certain hypothetical forty-five year term.

By substituting these values of Q in equation (12) the following values of the solar parallax result:

Moon's Mass.	$\frac{1}{80}$	$\frac{1}{81}$	$\frac{1}{82}$	$\frac{1}{83}$
$Q = 124''\cdot64$	$8''\cdot782$	$8''\cdot780$	$8''\cdot778$	$8''\cdot776$
125·36	·833	·831	·829	·827
125·46	·839	·837	·835	·833
125·64	·851	·849	·847	·845
126·46	8·910	8·908	8·906	8·904

These parallaxes are but little affected by the assumed mass of the moon, and depend almost entirely upon the observed value of the parallactic inequality, the relation between small changes of p and Q being

$$dp = 0\cdot071\, dQ \qquad (24)$$

The original observed values of Q are affected by personal equation, irradiation, blurring, and any error which may exist in the adopted semi-diameter of the moon. It is difficult to estimate how thoroughly these quantities are eliminated from the final result, but the remaining uncertainty probably amounts to a considerable fraction of a second.

Lunar Inequality of the Earth.—From observations at Greenwich, Paris and Kœnigsberg, made during the periods stated, LeVerrier found the following values for the lunar equation of the earth: (OPM, iv, 100)

Greenwich ----------	1816–26	$L = 6''\cdot45$
" ----------	1827–50	6·56
Paris --------------	1804–14	6·61
" --------------	1815–45	6·47
Kœnigsberg --------	1814–30	6·43

The mean is $6''\cdot50 \pm 0''\cdot023$.

Professor Newcomb found the following additional values: (WOb, 1865, App. II, pp. 25 and 26)

Greenwich -------	1851–64	$L = 6''\cdot56 \pm 0''\cdot04$
Washington ------	1861–65	$6\cdot51 \pm 0\cdot07$

With these values of L, equation (14) furnishes the following values of the solar parallax:

Moon's Mass.	$\frac{1}{80}$	$\frac{1}{81}$	$\frac{1}{82}$	$\frac{1}{83}$
$L = 6''\cdot50$	$8''\cdot664$	$8''\cdot770$	$8''\cdot878$	$8''\cdot985$
6·51	·678	·784	·892	8·999
6·56	8·744	8·851	8·960	9·068

It would seem that the observed value of L should be quite free from systematic errors, because it depends upon observations of the sun which are always made in the same way. The relation subsisting between small changes in the parallax, the mass of the moon, and the earth's lunar inequality, are given by the equation

$$dp = 1\cdot36\, dL + 0\cdot107\, d\left(\frac{1}{M}\right) \tag{25}$$

It will be difficult to determine the true value of L within $\pm 0''\cdot02$, and at present the uncertainty in the reciprocal of the moon's mass is at least $\pm 0\cdot5$. With these data the probable error of p comes out $\pm 0''\cdot06$.

Photo-tachymetrical Methods.

Theory.—The photo-tachymetrical methods are quite recent, having come into existence about 1850, when Fizeau and Foucault made their inventions for measuring the velocity with which light traverses moderate distances upon the surface of the earth. From the velocity of light thus obtained the solar parallax may be found by two essentially different methods, to wit:

1st. Deriving from the eclipses of Jupiter's satellites the time occupied by light in traversing the mean distance between the earth and the sun, and combining this with the measured velocity of light, we have

$$\tan p = \frac{\rho}{V\theta} \tag{26}$$

2d. Assuming the ratio of the earth's orbital velocity to the velocity of light to be represented by the constant of aberration, and combining that constant with the measured velocity of light, we have

$$\tan p = \frac{2\pi\rho}{TV \tan \alpha \sqrt{1-e^2_1}} \tag{27}$$

If p and V are eliminated between (26) and (27) we get

$$\tan \alpha = \frac{2\pi\theta}{T\sqrt{1-e^2_1}} \tag{28}$$

which shows the relation between α and θ.

For the constants in these equations I adopt

$\rho = 6378{\cdot}39$ kilometers (Col. Clarke's value).
T=31,558,149 seconds of mean time.
$e_1 = 0{\cdot}016771$

and the equations become

$$p = \frac{[9{\cdot}11914]}{\theta V} \tag{29}$$

$$p = \frac{[7{\cdot}73269]}{\alpha V} \tag{30}$$

$$\theta = [1{\cdot}38644]\alpha \tag{31}$$

the quantities within the square brackets being the logarithms of the numbers which they represent. In connection with equations (26), (27), (28), the reader may consult Cornu, OPM, t. xiii, pp. A 299–A 301.

Velocity of Light.—The following are the principal experimental determinations of the velocity of light between points upon the earth's surface:

	Kilometers.
1849. Fizeau (CRH, 1849, t. xxix, p. 90),	315,320
1862. Foucault (CRH, 1862, t. lv, p. 796: Recueil des travaux scientifiques de Léon Foucault, pp. 216–226),	298,000
1874. Cornu (OPM, xiii, 293),	300,400
1876. Helmert (ANn, 1876, bd. lxxxvii, s. 126),	299,990
1879. Michelson (Proc. Amer. Assoc., 1879, pp. 124–160),	299,940
1881. Young and Forbes (Nature, 1881, vol. xxiv, p. 303),	301,382

Light Equation.—The time taken by light to traverse the mean radius of the earth's orbit is commonly called the light equation, and there are but two determinations of it from the eclipses of Jupiter's satellites, namely:

1792. Delambre, from more than a thousand eclipses of the first satellite (Astronomie par Jerome le Français (la Lande), 3me edition. Paris, 1792, t. i, Tables astronomiques, p. 238. Also, Tables Ecliptiques des Satellites de Jupiter, par M. Delambre. Paris, 1817, p. vii, ---------- $493^s \cdot 2$

1874. Glasenapp (Investigation of the eclipses of Jupiter's satellites. A dissertation for the degree of master of astronomy, by S. Glasenapp. Published in the Russian language, at St. Petersburg, 1874, p. 131), ---------- 500·84

Glasenapp considered the probable error of his determination to be $\pm 1^s \cdot 02$.

Aberration.—The following are the principal determinations of the coefficient of aberration :

1728. Bradley (PTr, 1728, p. 655), ---------- 20″·25
1821. Brinkley (PTr, 1821, p. 350), ---------- 20·37
1840. Henderson (MAS, 1840, xi, 248), ---------- 20·41
1843. W. Struve (ANn, 1843, bd. xxi, s. 58), ---------- 20·445
1844. C. A. F. Peters (ANn, 1844, bd. xxii, s. 119), ---------- 20·503
1850. Maclear (MAS, 1851, vol. xx, p. 98), ---------- 20·53
1861. Main (MAS, 1861, vol. xxix, p. 190), ---------- 20·335

Solar Parallax.—The table below exhibits the various values of the solar parallax deducible from the foregoing values of V, θ and a by means of equations (29) and (30). I have rejected Fizeau and Foucault's values of the velocity of light on the ground that they are merely first approximations, the details of which have never been published; and I have made no use of Helmert's rediscussion of Cornu's value. The last column of the table gives the values of a and θ computed by means of equation (31) from the values of θ and a in the first column.

Velocity of Light ----	299,940	300,400	301,382	
Light equa'n				Aberration
493s·20	8″·894	8″·880	8″·851	20″·26
500·84	8·758	8·745	8·716	20·57
Aberration				Light equa'n
20″·25	8″·897	8″·883	8″·854	493s·02
·335	·860	·846	·817	495·09
·37	·844	·831	·802	495·94
·41	·827	·814	·785	496·91
·445	·812	·799	·770	497·76
·503	·787	·773	·745	499·19
20·53	8·775	8·762	8·734	499·84

The relations between small changes in p, θ, a, and V, are given by the equations

$$dp = -0{\cdot}0177\, d\theta - 0{\cdot}0295\, dV \quad (32)$$

$$dp = -0{\cdot}432\, d\alpha - 0{\cdot}0295\, dV \quad (33)$$

where θ is in seconds of mean time, a in seconds of arc, and V in thousands of kilometers. To determine p with a probable error not exceeding $\pm 0''{\cdot}01$, the probable errors of the other quantities must not exceed the following values, namely: θ, $\pm 0^{s}{\cdot}40$, and V, $\pm$ 240 kilometers; or a, $\pm 0''{\cdot}016$, and V, $\pm$ 240 kilometers. Whatever may be said respecting V, it is quite certain that our present knowledge of θ and a does not approach this degree of accuracy. The probable error of p seems to be at least $\pm 0''{\cdot}05$.

The photo-tachymetric method is embarrassed by serious theoretical difficulties. 1st. As we are ignorant of the optical constitution of inter-planetary space, we have no sure means of passing from the velocity of light at the earth's surface to its velocity in space. 2d. There is no rigorous proof that the constant of aberration gives the exact ratio of the velocity of light to the earth's orbital velocity. 3d. The velocity of light is the velocity of transmission of a single wave, while Fizeau's and Foucault's methods determine the velocity of transmission of a group of waves. Lord Rayleigh has shown that these two things are not necessarily the same. If the ordinary theory of aberration is accepted the velocity of light to which it refers is the velocity of a single wave, while the velocity determined from the eclipses of Jupiter's satellites is that of a group of waves. (Nature, 1881, vol. xxiv, pp. 382 and 460.)

Respecting the theory of aberration the reader may consult, Ann. de Chimie et de Physique, 1818, t. ix, p. 57; Oeuvres completes d'Augustin Fresnel, t. ii, p. 627; Stokes, in L. E. and D. Phil. Mag. 1845, vol. xxvii, p. 9; 1846, vol. xxviii, p. 76; 1846, vol. xxix, p. 6; Klinkerfues, in ANn, 1866, bd. lxvi, s. 337; 1868, bd. lxx, s. 239; 1870, bd. lxxvi, s. 33; Sohncke, ANn, 1867, bd. lxix, s. 209; Hoek, ANn, 1867, bd. lxx, s. 193; Veltmann, ANn, 1870, bd. lxxv, s. 145; Airy, Greenwich Observations, 1871, p. cxix; Proceed. Roy. Soc. 1873, vol. xxi, p. 121; Villarceau, Conn. de Temps, 1878, Additions; Michelson, this Journal, 1881, vol. xxii, p. 120.

Conclusion.

For convenience of reference the limiting values of the solar parallax, found by the various methods described in the foregoing pages, are presented here. It should be remarked, however, that in selecting these values the results of all discussions made prior to 1857 have been omitted; except in the case of the transit of 1761, and the smaller of the two values from the transit of 1769.

I.—Trigonometrical methods.	
Mars, meridian observations	8″·84 — 8″·96
" diurnal observations	8·60 — 8·79
Asteroids	8·76 — 8·88
Transit of Venus, 1761	8·49 — 10·10
" " 1769	8·55 — 8·91
" " 1874	8·76 — 8·85
II.—Gravitational methods.	
Mass of the earth	8″·87 ± 0″·07
Parallactic Inequality	8·78 — 8·91
Lunar Inequality	8·66 — 9·07
III.—Photo-tachymetrical methods.	
Velocity and Light Equation	8″·72 — 8″·89
Velocity and Aberration	8·73 — 8·90

To obtain a definitive value of the solar parallax, it would now be necessary to form equations of condition embodying the relations between the various elements involved; to weight these equations; and to solve for p by the method of least squares. But what is the use? It is perfectly evident that by adopting suitable weights, almost any value from 8″·8 to 8″·9 could be obtained; and no matter what the result actually was, it would always be open to a suspicion of having been cooked in the weighting. We only know that the parallax seems to lie between 8″·75 and 8″·90, and is probably about 8″·85. Attack the problem as we will, the results cluster around this central value. All the methods give a probable error of about ±0″·06, and no one of them seems to possess decided superiority over the others. We have nearly exhausted the powers of our instruments, and further advance can only be made at the cost of excessive labor.

In the beginning of the eighteenth century the uncertainty of the solar parallax was fully two seconds; now it is only about 0″·15. To narrow it still further, we require a better knowledge of the masses of the earth and moon, of the moon's parallactic inequality, of the lunar equation of the earth, of the constants of nutation and aberration, of the velocity of light, and of the light equation. All these investigations can be carried on at any time, but there are others equally important which can only be prosecuted when the planets come into the requisite positions. Among the latter are observations of Mars when in opposition at its least distance from the earth, and transits of Venus.

In 1874 all astronomers hoped and believed that the transit of Venus which occurred in December of that year would give the solar parallax within $0''{\cdot}01$. These hopes were doomed to disappointment, and now, when we are approaching the second transit of the pair, there is less enthusiasm than there was eight years ago. Nevertheless, the astronomers of the twentieth century will not hold us guiltless if we neglect in any respect the transit of 1882. Observations of contacts will doubtless be made in abundance, but our efforts should not cease with them. We have seen that the probable error of a contact observation is $\pm 0''{\cdot}15$, that there may always be a doubt as to the phase observed, and that a passing cloud may cause the loss of the transit. On the other hand, the photographic method cannot be defeated by passing clouds, is not liable to any uncertainty of interpretation, seems to be free from systematic errors, and is so accurate that the result from a single negative has a probable error of only $\pm 0''{\cdot}55$. If the sun is visible for so much as fifteen minutes during the whole transit, thirty-two negatives can be taken, and they will give as accurate a result as the observation of both internal contacts. In view of these facts, can it be doubted that the photographic method offers as much accuracy as the contact method, and many more chances of success?

The transit of 1882 will not settle the value of the solar parallax, but it will contribute to that result, directly as a trigonometrical method, and indirectly through the gravitational methods with which the final solution of the problem must rest. As our knowledge of the earth's mass may be made to depend upon quantities which continually increase with the time, it will ultimately attain great exactness, and then the solar parallax will be known with the same exactness. Long before that happy day arrives the present generation of astronomers will have passed over to the silent majority, but not without the satisfaction of knowing that their labors will contribute to that fullness of knowledge which shall be the heritage of their successors.

Washington, D. C., October, 1881.