

ART. LIX.—*A Historical Note on the Method of Least Squares ;*
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It is well known that the "Method of Least Squares," although first published in printed form by LeGendre in 1806, in his "Nouvelles Methodes," was first invented by Gauss as early as 1795, and had for years been taught by him in his lectures to his students at Göttingen. It was, however, some years before the Gaussian method came into general use, and especially were English scholars very slow to acquaint themselves with its merits. I have, therefore, been much interested in finding that, in 1808, Professor Robert Adrain, at that time in New Brunswick, N. J., published the method of least squares in the "Analyst," having been independently led to this invention by the study of a prize problem offered some months previously in that periodical.

As the editor of, and chief contributor to, the Mathematical Correspondent, the Analyst, and the Mathematical Diary, and

as Professor in Columbia College and in Pennsylvania University, as well as by his correspondence, Dr. Adrain is well known to have contributed powerfully to the progress of Mathematical studies in this his adopted country—(he was born and educated in Dublin)—and his apparently independent demonstration of the method of least squares seems quite in accordance with the originality shown in many other of the elegant solutions offered by him to the different problems on which he busied himself. A number of interesting and probably valuable mathematical manuscripts still remain in the possession of his family at New Brunswick, New Jersey, which it is to be hoped may some day see the light. At present I would offer toward the history of mathematics in America the following extracts from the *Analyst* and other publications.

The problem “to correct the distances and bearings of a survey, so as to deduce the most probable area of the enclosed field,” had been proposed by Professor Patterson in a previous number of the *Analyst*, and after being a second time renewed as a prize question, was at length in number IV, solved by a course of special reasoning, by Dr. Bowditch, to whom Dr. Adrain awarded the prize. Dr. Bowditch’s results coincided with what would have been deduced had the Gaussian method been applied to this case. Immediately following Dr. Bowditch’s special solution, the editor adds his own solution of the more general problem as follows: (The *Analyst*, pp. 93–95 inclusive).

“*Research concerning the probabilities of the errors which happen in making observations.*”

“The question which I propose to resolve is this: supposing AB to be the true value of any quantity of which the measure by observation or experiment is Ab , the error being Bb ; what is the expression of the probability that the error Bb happens in measuring AB?”

Let AB, BC, &c., be several successive distances of which the values by measure are Ab , bc , &c., the whole error being Cc ; now supposing the measures Ab , bc , to be given and also the whole error Cc , we assume as a self-evident principle, that the most probable distances AB, BC are proportional to the measures Ab , bc ; and therefore the errors belonging to AB, BC are proportional to their lengths, or to their measured values Ab , bc . If therefore we represent the values of AB, BC or of their measures Ab , bc by a , b , the whole error Cc by C , and the errors of the measures Ab , bc by x , y , we must for the greatest probability, have the equation $\frac{x}{a} = \frac{y}{b}$. Let X and Y be similar functions of a , x , and of b , y , expressing the probabilities that the errors x , y happen in the distances a , b ; and, by the

fundamental principle of the doctrine of chance, the probability that both these errors happen together will be expressed by the product XY . If now we were to determine the values of x and y from the equations $x+y=E$ and $XY=\text{maximum}$, we ought evidently to arrive at the equation $\frac{x}{a}=\frac{y}{b}$: and since x and y are rational functions of the simplest order possible of a, b and E , we ought to arrive at the equation $\frac{x}{a}=\frac{y}{b}$ without the intervention of roots, in other words by simple equations; or, which amounts to the same thing in effect, if there be several forms of X and Y that will fulfill the required condition we must choose the simplest possible, as having the greatest possible degree of probability.

“Let X', Y' be the logarithms of X and Y , to any base or modulus; and when $XY = \text{max.}$ its logarithm $X' + Y' = \text{max.}$ and therefore $\dot{X}' + \dot{Y}' = 0$, which fluxional equation we may express by $X''\dot{x} + Y''\dot{y} = 0$; for as X' involves only the variable quantity x , its fluxion \dot{X}' will evidently involve only the fluxion of x ; in like manner the fluxion of Y' may be expressed by $Y''\dot{y}$; and from the equation $X''\dot{x} + Y''\dot{y} = 0$ we have $X''\dot{x} = -Y''\dot{y}$: but since $x + y = E$ we have also $\dot{x} + \dot{y} = 0$, and $\dot{x} = -\dot{y}$, by which dividing the equation $X''\dot{x} = -Y''\dot{y}$, we obtain $X'' = Y''$.

“Now this equation ought to be equivalent to $\frac{x}{a} = \frac{y}{b}$; and this circumstance is effected in the simplest manner possible, by assuming $X'' = \frac{mx}{a}$, and $Y'' = \frac{my}{b}$; m being any fixed number which the question may require.

“Since, therefore, $X'' = \frac{mx}{a}$ we have $X''\dot{x} = \dot{X}' = \frac{m\dot{x}x}{a}$, and taking the fluent, we have $X' = a' + \frac{mx^2}{2a}$. The constant quantity a' being either absolute, or some function of the distance a .

“We have discovered, therefore, that the logarithm of the probability that the error x happens in the distance a is expressed by $a' + \frac{mx^2}{2a} = X'$, and consequently the probability it-

self is $X = e^{X'} = e^{\left(a' + \frac{mx^2}{2a}\right)}$. Such is the formula by which the probabilities of different errors may be compared, when the values of the determinate quantities e, a' and m are properly adjusted. If this probability of the error x be denoted

by u , the ordinate of a curve to the abscissa x , we shall have
 $u = e^{\left(a' + \frac{mx^2}{2a}\right)}$, which is the general equation of the curve of probability.

“When only the maximum of probability is required, we have no need of the values of e , a' and m ; it is proper, however, to observe that m must be negative. This is easily shown. The probability that the errors x , y , z , etc., happen in the dis-

tances a , b , c , etc., is $e^{\left(a' + \frac{mx^2}{2a}\right)} \times e^{\left(b' + \frac{my^2}{2b}\right)} \times e^{\left(c' + \frac{mz^2}{2c}\right)}$, etc.,

which is equal to $e^{\left(a' + b' + c', \text{ etc.}, + \frac{mx^2}{2a} + \frac{my^2}{2b} + \frac{mz^2}{2c}, \text{ etc.}\right)}$, and this quantity will evidently be a maximum or minimum as its index or logarithm is a maximum or minimum; that is, when

$$\frac{m}{2} \left\{ \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c}, \text{ etc.}, \right\} = \text{a maximum or minimum.}$$

Now when $x + y + z$, etc., = E , we know that

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c}, \text{ etc.}, = \text{minimum, when } \frac{x}{a} = \frac{y}{b} = \frac{z}{c}, \text{ etc.},$$

and therefore $-\left\{ \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c}, \text{ etc.}, \right\} = \text{maximum.}$

When $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, etc., it is evident therefore that m must be negative; and as we may for the case of maxima use any value of it we please, we may put $m = -2$, and the probability of x in a is $u = e^{\left(a' - \frac{x^2}{a}\right)}$. If we put $\frac{m}{2a} = -1$ and $a' = f^2$, we have $u = e^{(f^2 - x^2)}$ for the equation of the curve of probability; but if we suppose $f^2 = 0$, the ordinates u will still be proportional to their former values, and we shall have $u = e^{-x^2}$, or $u = \frac{1}{e^{x^2}}$, which is the simplest form of the equation expressing the nature of the curve of probability.”

Immediately following the above general solution by Dr. Adrian there are given applications of this method to the following problems.

1. To find the most probable value of any quantity of which a number of direct measures are given.
2. To find a most probable position of a point in space.
3. To correct in the most probable manner the dead reckoning at sea.
4. To correct the bearing and distances of a field survey.

The article closes on p. 109 of the *Analyst* with the following:
 "I have applied the principles of this essay to the determination of the most probable value of the earth's ellipticity, &c., but want of room will not permit me to give the investigation at this time."

The investigations here alluded to were, however, long afterwards published, i. e., in 1817, in vol. i, new series, of the *Transactions of the American Philosophical Society*, and are given in two papers (Nos. IV and XXVIII) of that volume. The preceding note as well as the dates written on the manuscripts (which are still preserved by the Hon. G. B. Adrian of New Brunswick) show that these two investigations were completed in 1808.

The first of the papers here alluded to is entitled "Investigation of the figure of the earth and of the gravity in different latitudes," from which as printed in the *Phil. Trans.*, we make the following extract:

"Having in the year 1808 discovered a general method of resolving several useful problems by ascertaining the highest degree of probability, when certainty cannot be found, I shall here apply that method to the determining of the earth's ellipticity, &c." The author's computation is based on the lengths of the seconds pendulum as given by Laplace (*Mec. Cel.*, iii), and having stated the problem before him, he says: "This is accomplished by a rule published by the writer in the *Analyst*, in 1808." The resulting ellipticity ($\frac{1}{3} \frac{1}{7}$) he shows to differ from that deduced by Laplace ($\frac{1}{3} \frac{1}{7}$) because of numerical errors in the computation of the latter; having corrected these he deduces the ellipticity $\frac{1}{3} \frac{1}{7} \cdot \frac{1}{5}$ by Laplace's own method—showing that the two methods conduce to nearly the same result.

The second of the articles in the *Phil. Trans.*, is entitled "A Research concerning the Mean Diameter of the Earth." In this the author seeks the sphere which most nearly coincides in various specified peculiarities with the actual terrestrial spheroid; the diameter of this sphere he determines to be 7918·7 miles. This numerical result is based upon some earlier computations, the details of which are not given, but of which he says: "Having determined the most probable axis of the terrestrial spheroid from the measurements of a degree of the meridian by a method which I discovered several years ago and published in the *Analyst*, the resulting mean radius was found to be 3959·69 English miles."

The mathematical works published by Dr. Adrian are so rarely to be met with, that it was necessary to make these long extracts in order to establish the conclusion to which we have arrived, i. e., that we must credit Dr. Adrian with the independent invention and application of the most valuable arithmetical process that has been invoked to aid the progress of the exact sciences.

Washington, Feb. 22, 1871,